# RESEARCH REPORT

NATIONAL CENTER FOR IMPROVING STUDENT LEARNING
AND ACHIEVEMENT IN MATHEMATICS AND SCIENCE

# DEVELOPING CONCEPTIONS OF ALGEBRAIC REASONING IN THE PRIMARY GRADES

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# Developing Conceptions of Algebraic Reasoning in the Primary Grades

There is a substantial body of research describing the development of children's informal mathematical thinking and how this knowledge can serve as a basis for developing understanding of basic concepts and procedures of arithmetic (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Franke, M.L., Jacobs, V., Fennema, 1998; Hiebert et al., 1997). The goal of the research described below is to investigate how to design instruction to help children to take the next step to generalize and formalize their knowledge into powerful abstract systems for representing and operating on mathematical ideas (i.e. to move from arithmetic thinking to algebraic thinking). The studies build on existing research on Cognitively Guided Instruction (CGI; Carpenter, Fennema, & Franke, 1996; Fennema, et al., 1996). Children in CGI classrooms construct a variety of increasingly sophisticated and abstract strategies for solving a range of problems involving the basic operations of arithmetic, and they use these strategies flexibly and creatively in solving problems. The strategies emerge out of and are related to the problem situations that give them meaning. Our goal in the studies reported in this paper is to begin to understand how to provide support for children to reflect on their procedures in order to form generalizations from them and construct notations for representing their procedures and generalizations abstractly.

## Background

Kaput (in press) and others have argued that developing algebraic reasoning at the elementary level is critical for reforming mathematics teaching. Kaput proposes that algebra reform is at the center of mathematics curriculum reform and that the ability of elementary teachers to develop algebraic reasoning may be the most critical factor for algebra reform and hence for reform of the mathematics curriculum in general. There is an emerging consensus, however, that it is necessary to reconceptualize the nature of algebra and algebraic thinking and to reexamine when children are capable of algebraic thinking and when ideas that require algebraic thinking should be introduced into the curriculum (Kaput, in press; NCTM, 1993, 1994, 1997, 1998). It is proposed that the artificial separation of arithmetic and algebra deprives students of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades. Understanding takes a long time to develop, and algebraic thinking is conceived as developing over an extended period of time starting in the early elementary grades. This does not mean, however, simply pushing the current high school algebra curriculum down into the elementary school. There is a broader conception on the nature of algebra in which the emphasis is not on learning rules for manipulating symbols. The goal is to develop algebraic thinking, not skilled use of algebra procedures.

Two central themes are at the core of this new conception of algebraic thinking: (a) making generalizations and (b) using symbols to represent mathematical ideas and to represent and solve problems. Students are encouraged to make explicit powerful unifying ideas and to construct ways to represent those ideas for thinking about them and for communicating them. Simple examples at the primary grades include such generalizations as "When you add zero to a number, the sum is always that number" or "When you add three numbers, it does not matter which two you add first." Young children are capable of making such generalizations and constructing ways of representing them (Bastable & Schifter, in press; Davis, 1964 ; Kaput, in press; Schifter, 1999; Tierney & Monk, in press). These generalizations make powerful mathematical ideas accessible to students to solve problems and to deepen understanding. Generalization and formalization involve the articulation and representation of unifying ideas that make explicit important mathematical relationships. Thus, these forms of thinking build directly on conceptions of understanding as constructing relationships and reflecting on and articulating those relationships (Carpenter & Lehrer, 1999). In fact they can be viewed as an attempt to further articulate how the development of understanding is instantiated in the elementary grades.

## BUILDING ON CHILDREN'S ARITHMETICAL THINKING

Our research builds directly on our previous work on Cognitively Guided Instruction, which focused on the development of children's arithmetical thinking (Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1996; Fennema et al.,1996). We are investigating how the kinds of meaning making that we see in classrooms in which teacher's decisions are guided by their knowledge of children's thinking can be extended to encompass the forms of generalization and formalization that characterize algebraic thinking. Children in CGI classrooms construct a variety of increasingly sophisticated and abstract strategies for solving a range of problems involving the basic operations of arithmetic, and they use these strategies flexibly and creatively in solving problems. Children initially solve problems by modeling the problem situations using physical materials. By reflecting on the modeling strategies, children abstract these strategies so that they no longer need the actual materials to solve the problem. They use computational procedures that are essentially abstractions of the physical manipulations they initially used to solve the problems. The next extension involves reflecting on the computational procedures themselves. Our goal in this project is to understand how to provide support for children to reflect on their procedures in order to form generalizations from them and construct notations for representing them abstractly.

Schifter (1999) argued that algebraic thinking is implicit in the kind of meaning-making activities and invented strategies for solving problems that are the mainstay of mathematical activity in CGI classrooms. As children construct strategies for solving problems, they draw upon fundamental properties of addition, subtraction, multiplication, and division and the relations among the operations. For example, some solutions for Join Change Unknown (or missing addend) problems depend on understanding the inverse relation between addition and subtraction, and counting-on-from-larger strategies depend on commutativity of addition (Carpenter, Fennema, & Franke, 1996). It is a large step, however, to make explicit these fundamental understandings about number properties. Children who might readily use commutativity to solve problems with small numbers might express doubts whether the property holds for all numbers (Bastable & Schifter, in press). On the other hand, Bastable and Schifter (in press), Davis (1964), Tierney and Monk (in press), and others have demonstrated that when elementary age children are provided the opportunity they are capable of formulating and verifying generalizations about number operations.

Teachers working with Bastable and Schifter (in press) describe a number of cases in which generalizations about fundamental number properties arose spontaneously in their classes as students were solving problems. Often a strategy that a student used triggered a discussion of whether the strategy always worked and subsequently what that meant in terms of more general statements about a given operation. Building on students' spontaneous observations about regularities in number patterns and relationships in problem-solving strategies offer one context in which students can articulate generalizations about number properties, but in our early pilot work we found that the opportunities did not occur with great frequency, and it was difficult to initiate them if they did not arise spontaneously. In order to provide a context to initiate conversations that could lead to generalization and to introduce discussion of notation that might be used to express those generalizations, we drew most directly on Bob Davis's (1964) work on the Madison Project, in particular his activities involving true-false and open number sentences.

Although we looked for examples of students' algebraic thinking in all of their work, the primary context in which we have engaged students in activities that might lead to algebraic thinking was through the solution and discussion of true-false and open number sentences. The students in the two studies reported in this paper verified the truth of true-false number sentences ($23 - 14 = 9$, $7 + 8 = 13$, $67 + 54 = 571$) and solved open number sentences of a variety of forms. ($x$[1] $+ 57 = 84$, $x + y = 7$, $x + x = 24$). At appropriate times, number sentences and groups of number sentences were selected to highlight and provide a context for discussion of basic properties of numerical operations and relations. For example, verifying the truth of $567 + 0 = 567$ could naturally lead students to articulate generalizations about zero.

Students studying algebra often think of variables as representing a specific number that is the solution to an equation. This can create some problems in the ways students conceive of variables in other situations like using variables to describe properties of numbers ($x + y = y + x$). Starting with true-false number sentences and relating open number sentences to them potentially provides a more unified conception of variable. Questions such as "Can you think of an open number sentence that is true no matter what numbers you put in for the variables?" can provide students a door to using variables to express generalizations (Davis, 1964).

## *Study 1*

In the spring of 1997, we taught a series of lessons to a group of eight of the students in a combination first- and second-grade class taught by Annie Keith, an experienced CGI teacher—

[1]In the initial study, we generally used boxes, triangles, and other shapes to represent variables. In the second study, we found that children readily use letters for variables, and they were commonly used by the teacher and students in that study. We use letters in this paper for simplicity unless the specific notation is of importance.

who we have worked with and studied for over 10 years. The group included four girls and four boys: One girl was a first grader; the other seven students were second graders. We taught eight lessons over a period of about a month. Mathematics instruction in class was almost always based on word problems, but the students regularly wrote open number sentences for problems and were familiar with standard notation for addition and subtraction. The class routines included regular discussion of alternative strategies for solving problems, so the students were used to talking about how they solved problems; in fact they expected to describe their strategies. They were not used to discussing arithmetic sentences that were not put in context, they generally were not used to talking about properties of operations, and they had not encountered true-false number sentences or open number sentences involving multiple variables or repeated use of a single variable.

We started with true-false number sentences. Although the students had no experience with true and false number sentences, they readily verified whether given sentences were true or false. In the sessions that followed, we moved to open number sentences with one variable, open number sentences with two variables, and finally to open number sentences with a repeated variable. The students quite readily found the number or numbers that would make given number sentences true for all of the different forms.

A major conceptual issue related to the development of algebraic thinking is the notion of equality as a relation. Behr, Erlwanger, and Nichols (1975) and Erlwanger & Berlanger (1983) documented that children in the elementary grades generally consider that the equal sign means to do something, to carry out the calculation that precedes it, and that the number after the equal sign is the answer to the calculation. This is how the notation is usually used in the elementary school, but it is one of the major stumbling blocks when moving from arithmetic to algebra (Kieran, 1981; Matz, 1982). True-false and open number sentences offered an opportunity to challenge children's incorrect notions of meaning of the equal sign before they became entrenched. The first day of the study, we provided children with true-false number sentences that challenged their conceptions of the meaning of the equal sign. We started with a variety of true-false number sentences written in a familiar form with a single number after the equal sign. After the children responded to about a dozen problems of this type, we presented them with a number sentence like $4 + 3 = 5 + 2$. Their initial reaction was that we could not write a number sentence like that. We told them that it was acceptable to write number sentences like that, and challenged them to decide whether the sentences could be true. After some discussion, they came to accept number sentences in which the "answer" did not immediately follow the equal sign. By the end of the study, they readily worked with number sentences that involved thinking of the equal sign expressing a relation.

One of our primary goals was to use number sentences as a context to engage the students in a discussion of properties of number operations to assess if they could make explicit their implicit generalizations about number operations and what rules of evidence they would use to support their generalizations. The first properties we considered involved operations with zero. The children immediately recognized that $58 + 0 = 58$ was a true sentence. Rather than immediately pressing them to make a generalization, we gave them the following number sentence: 78 – 49 = 78. The following interaction occurred:

Children: "False!" "No, no false!" "No way!"

Teacher: Why is that false?

Jenny: Because it is the same number as in the beginning, and you already took away some, so it would have to be lower than the number you started with.

Mike: Unless it was 78 – 0 = 78. That would be right.

Teacher: Is that true? Why is that true? We took something away.

Steve: But that something is, there is, like, nothing. Zero is nothing.

Teacher: Is that always going to work?

Lynn: If you want to start with a number and end with a number, and you do a number sentence, you should always put a zero. Since you wrote 78 – 49 = 78, you have to change a 49 to a zero to equal 78, because if you want the same answer as the first number and the last number, you have to make a zero in between.

Teacher: So do you think that will always work with zero?

Mike interpreted the question as whether it was necessary to change the 49 to a zero:

Mike: Oh no. Unless you 78 minus, umm, 49, plus something.

Ellen: Plus 49?

Mike: Yeah. 49. 78 – 49 + 49 = 78.

Teacher: Wow. Do you all think that is true? [All but one child answers yes.]

Jenny: I do, because you took the 49 away, and it's just like getting it back.

Essentially the children generated another generalization ($a + b - b = a$), although they had not yet articulated it as a general rule. It is a somewhat more difficult generalization to articulate than the zero properties for addition and subtraction, so after some discussion of the specific example, we returned to sums and differences involving zero with the following example: 789,564 – 0 = 789,564.

Children: That's true.

Teacher: How do you know that is true? Have you ever done that? Ann?

Ann: I will tell you. All those numbers take away zero, you won't take away anything, so it would be the same number.

After another example in which the children immediately respond that 0 + 5869 = 5869 was true, the following discussion ensued:

Teacher: So we kind of have a rule here, don't we? What's the rule?

Ann: Anything with a zero can be the right answer.

Mike: No. Because if it was 100 + 100 that's 200.

Jenny: That's not that we are talking about. It doesn't have just plain zero.

Ann: I said, umm, if you have a zero in it, it can't be, like, 100 because you want just plain zero, like 0 + 7 = 7.

After some additional discussion to clarify that the children were talking about the number zero, not zero in numbers like 20 or 500, the children were challenged to state a rule that they could share with the rest of the class.

Ellen: When you put zero with one other number, just one zero with the other number, it equals the other number.

Steve: Not true.

Teacher: Wait. Let me make sure I got it. You said, "If you have a plain zero with another number." With another number? Like just sitting next to the number?

Ellen: No, added with another number, or minus from another number, it equals that number.

The group collectively came up with the rule: "Zero added with another number equals that other number." They also came up with the generalizations: "Zero subtracted from another number equals that number," and "Any number minus the same number equals zero." One student, Steve, came up with several generalizations about multiplication. The comments came up in response to Ellen's initial generalization about zero, which did not specify an operation.

Steve: I wasn't thinking about the zero stuff plus another number equals that same number that we added to the zero, but, umm, I was thinking about if you were [inaudible] a number times zero would be zero. . . . 7 x 0 = 0. That is what I am trying to say, because 7 zeros or 0 sevens would be zero, and if you just add 7 zeros, you would just get zero. . . .Even a high number times zero would be zero... . Even the highest number you can think of times zero would be zero.

Teacher: How do you know that, Steve?

Steve: . . . Like 256 times zero, 256 zeros and any amount of zeros would be zero.

Steve had initially objected to Ellen's overly general statement of a zero rule. Having clarified the rules, we questioned Steve about whether his rule was an exception to the rules for addition and subtraction. He responded that those rules were like 27 x 1 = 27. Although we were unable to pursue this comment in depth because we were running out of time and the other children generally were not able to participate in the conversation, Steve's comment suggested that he had at least some level of understanding of the parallels between additive and multiplicative identities.

In these examples, children readily applied generalizations about zero to determine the truth value of number sentences. They not only applied them to solve given problems involving zeros; they came up with number sentences that embodied additional principles (78 – 49 + 49 = 78), and one student, Steve, spontaneously came up with generalizations that did not evolve out of solutions of specific number sentences. Children often tried to state their generalizations using a specific example ("It's like 7 + 0 = 7"), and they often used specific simple cases to validate their generalizations. They all were confident, however, that their generalizations held for all numbers. The justification of their assertions generally referred to zero representing "nothing," but they did say that zero was a number. Ellen's initial statement of the zero principle was overly general and not accurate, but collectively the students identified the limitations and constructed more focused, valid generalizations. At this point the generalizations were stated in natural language, which could be awkward and imprecise.

Another feature of the discussion was that the students seemed to demonstrate a good conception of the appropriate use of counterexamples to challenge other students' claims or generalizations. For example, Mike challenged Ann's generalization that "Anything with a zero can be the right answer" with the counterexample 100 + 100 = 200. That forced Ann to revise her assertion to make it more precise. Steve also challenged Ellen's generalization about operations with zero by bringing in multiplication. The students had a little more difficulty articulating why their assertions were true, but they generally seemed to recognize that a single case or several cases did not prove a statement was always true, and they attempted to use arguments that would apply to all numbers (e.g., "All those numbers take away zero, you won't take away anything, so it would be the same number," "Any amount of zeros would be zero").

## OPEN NUMBER SENTENCES

Over the next five sessions we introduced a variety of open number sentences. Although generalizations about number properties did come up in solving some open number sentences ( 78 + □€ = 78), we did not focus again on specifically articulating generalizations until the next to the last session. The session started with a group of four of the students solving a problem that one of the students had thought up (€□ + □ + □ – □€ 10). Two of the students described solutions based on substituting numbers for □, but Steve used a more principled strategy:

Steve: Well, since I would be adding on one number and then taking away the same number, I just kinda crossed out those, and then I thought 5 + 5 = 10.

[. . .]

Lynn: Well, I don't know if I exactly get what Steve is saying, but I think what he is saying is that if 5 + 5 is 10, and you just add the 5, you add another 5, you take it away, so no matter what, so it's not going to be there even if you didn't add it on.

Steve: It would be like you never added it on.

Lynn: Because it would equal zero, and then just to 5 + 5, because the third 5 wouldn't be added on, because it would be added on and then just taken away.

Essentially, the children in this episode implicitly employed two fundamental principles of number operations ($x - x = 0$, and $y + 0 = y$). Later in the session, they made the principles involved in this solution more explicit using variables.

The children also solved the equation □ + □ = Δ. They generated a number of solutions and recognized that Δ had to be twice as large as □. They discussed that although any number could be substitute for □, once □ was chosen, only one number could be substituted for Δ to make the sentence true. Thus, although there was an infinite number of pairs that could be chosen, not all substitutions for □€and Δ would result in a true sentence.

We then posed the following problem: "Can you write an open number sentence that is true for every number no matter what you put in?" It took a little time to clarify the task. One student initially suggested 1 + 1 = 2, which was not an open number sentence, and one student suggested □ + Δ = ❍, which is true for some substitutions, but not for all. Once the task was clarified, the children started to suggest possibilities like □€+ □€= □€but other children offered numbers that showed that the sentences were not true for all substitutions. After they had tried and rejected a number of suggestions, we provided some scaffolding by asking if there were any special numbers that might work. After a few tries Lynn came up with 0 + 0 + 0 + □€= □€ Steve and Ellen immediately picked up on what Lynn had done and excitedly pursued generating their own cases. Ellen came up with 0 + □€+ □€– □€= □€ This example invokes the same principle as the problem that the students had solved at the start of the lesson and is related to the true number sentence (78 – 49 + 49 = 78) that Mike and Ellen had generated in the first session. The students also generated the following number sentences: €□ + Δ– Δ = □€□€– □€= 0, 0 + □€+ 0 = □€

Steve suggested □€□€= 1, which he justified by arguing that two cookies shared among two people would give each person a cookie. He said the same thing would happen with 4, but then he started to think how he would explain negative numbers and zero, but he could not resolve this. Because all of the solutions involved properties of zero except Steve's solution, which involved a parallel multiplicative principle, we presented the following problem: □€+ Δ= □€+ Δ. The children all agreed that this number sentence was true for all numbers, and that prompted Ellen to come up with an open sentence representing the commutative property: □€+ Δ= □€+ Δ€ All of the students agreed that it was true for all numbers, but their justifications were based on specific cases. When we asked how they would convince someone else that it was always true, they gave examples and said that they would ask the person to come up with a case for which it was not true. They did not seem completely satisfied with this answer, but they could not then generate a more general explanation for commutativity.

## STUDY 1 CONCLUSIONS

The study demonstrated that some first- and second-grade children could deal successfully with a variety of true-false and open number sentences and that these sentences could provide a forum for focusing discussions on generalizations and a notational system with potential for expressing those generalizations. The school year ended soon after our experience writing open number sentences to express generalizations about numbers, and there were a number of questions still to be answered about children's conceptions of the notation and its potential for representing problems and expressing generalizations. The study demonstrated that number sentences could provide a context for productive mathematical activity that involved students in algebraic thinking as outlined at the beginning of this paper, but questions remained about ways to integrate that context into instruction in a regular classroom and the kind of interactions that would support the development of increasingly sophisticated generalizations and argumentation in support of these generalizations. We had worked with a small select group of students, and it was unclear how accessible these ideas would be to a more representative group of students. Those questions were addressed in Study 2.

## *Study 2*

The following year we conducted a case study of Annie Keith's combination first- and second-grade class of 20 students. The students in Study 1 had been in Ms. Keith's class the preceding year, and we shared the protocols of the lessons with her and engaged in extended discussions of the lessons with her in preparation for this study. During the year, we met regularly with Ms. Keith and several other teachers to help develop hypotheses about students' ability to generalize about fundamental arithmetic principles and relations and their ability to relate their generalizations to number sentences.

Ms. Keith taught algebra twice a week. Each week we observed the algebra instruction and took detailed field notes on students' solutions to problems and their interactions about them with the teacher and with each other. Near the end of the year, we individually interviewed each student for whom we had parental consent regarding their conceptions about their conceptions of the meaning of the equal sign, their ability to use open number sentences to express mathematical ideas, and their ability to use open number sentences to represent problems.

Instruction in the class followed somewhat the same pattern as in Study 1, starting with true-false number sentences and subsequently introducing increasingly complex forms of open sentences. Students' ability to recognize generalizations represented in true-false and open number sentences followed a pattern similar to that observed in Study 1.

### EQUALITY

The first important construct that the class addressed was that the equal sign expresses a relation. In early October, Ms. Keith presented the following number sentences to the children and asked them if they were true or false: $1 + 1 = 2$, $2 = 1 + 1$, $2 = 2$, and $1 + 1 = 1 + 1$. Although all the children were confident that the first number sentence was true, many struggled with the remaining number sentences. For example, when Ms. Keith asked Carla if $2 = 1 + 1$ was true, Carla replied, "I am not sure, it's backwards, the one plus one is on the other side, but it makes sense, so probably yes." By the end of the lesson, most children had agreed that all four number sentences were true and subsequently used the equal sign as expressing a relation throughout the next two months.

In the following January and February, most of the number sentences Ms. Keith posed to her students had a single number after the equal sign. Ms. Keith was addressing other issues and inadvertently started posing number sentences to her students with a single number after the equal sign. In March, Ms. Keith again posed a problem that required understanding that the equal sign expresses a relation. The class was discussing the following number sentences: 50 + 50 = 99 + 1 and 50 + 50 = 100 – 7. Children's responses to these problems indicated that some of the children's understanding of the equal sign was fragile. At least five children who had at one time used the equal sign consistently in a correct manner said that the first number sentence was false and the second one was true. Alex, for example, said, "50 + 50 is 100 not 99, so the top one is false." Two children then replied in a manner similar to Jae Meen who said, "Equal sign means they are the same. This side is 100, but this side is 93, so this one [50 + 50 = 100 – 7] is false." After further discussion, they came to the consensus that 50 + 50 = 99 + 1 is true and 50 + 50 = 100 – 7 is false and continued to show appropriate understanding of the equal sign throughout the school year. After this episode, Ms. Keith varied the location of the equal sign when presenting number sentences to her students.

## CONJECTURES AND JUSTIFICATION

As was the case for developing ideas of the meaning of the equal sign, Ms. Keith also used number sentences to help children articulate conjectures about properties of numbers and operations. The first time the children made a conjecture was in the middle of October. Ms. Keith posed the number sentence 1 – 100 = 5 and asked the children if it was true or false. When the class answered in chorus that it was false, the following discussion took place:

Ms. Keith: How did you know right away that this is false?

Kate: You can't take away 100. Since 1 is smaller than 100, you can only take away one.

Mary: You can't take a bigger number from a smaller one.

Mitch: It would be in the negatives…

Donald: If you take away a number, you have to get a lower number for your answer

[….]

Ms. Keith: Will that always be true?

Laura: Yes, when you subtract a number bigger than your starting number, you will always get a negative number for your answer.

Ms. Keith: Laura, I want us all to think about that, so I am writing it on the board.

After Ms. Keith engaged more children in this discussion. She asked them if they thought whether what Laura had said would always be true. Eleven children agreed; the four remaining children said they weren't sure. Ms. Keith then said that they would write this idea down and call it a *conjecture*. A conjecture was defined as a math idea that they thought was always true.

As was the case for the above conjecture, true-false number sentences provided the impetus for other conjectures. In mid-November, Ms. Keith had students write true-false number sentences with zero in them. Discussing these number sentences caused the children to generate conjectures about adding zero to a number and subtracting a number from itself. In several cases, the children initially proposed a conjecture that was incorrect or imprecise, and the class edited the conjecture as a group. For example, Carla proposed the following conjectures after reflecting on the number sentences 11 – 11 = 0, 13 – 13 = 0, and 200 – 200 = 0: "If the second number is the same number as the first number, it will always equal zero." The class worked on this conjecture, editing it to make it more precise. Dan said, "It won't always be zero if there is a plus there instead of a minus." Eventually the conjecture was stated as "If you subtract the same number from the same number, you will get zero."

By the end of the year the class had generated the following conjectures:

1. When you subtract a number bigger than your starting number, you will always get a negative number for your answer.
2. Zero plus a number equals that number.
3. If you subtract the same number from the same number, you will get zero.
4. If you subtract zero from a number, you will end up with the same number.
5. If you plus two identical whole numbers that are higher than zero, you'll get an even number.
6. If you add two odd whole numbers, you will get an even number.
7. If you add an even and an odd whole number, you will always get an odd whole number.

Each time the children generated a new conjecture, it was written on a piece of paper and taped to the wall.

Although the children were successful at generating conjectures, they had limited understanding of what was required to justify a conjecture. When children started making conjectures, Ms. Keith explored the possibility of their being able to justify these conjectures. She often asked children questions such as, "How do you know this will always be true?" In the following episode in October, the class discussed how they could decide that something is always true. At this time, the term conjecture had not been introduced; they were using the word *rule* to refer to what they eventually call a *conjecture*. The word *conjecture* was introduced later in the lesson.

Ms. Keith: How will we decide that a rule is always true?

Kate: When we all agree on it.

Laura: I think when we all try it and it works.

Ms. Keith: So you are thinking that maybe we can try it with some different numbers?

Laura: Yes.

[….]

Observer: What if we find one time that it's not true, can it still be a rule?

[Some children said no.]

Observer: What if there is only one time that it is not true, can it still be a rule?

[Again, some children said no.]

Mitch: Well, yeah, it could still be a rule.

John: A false rule though.

Sam: It can't be a rule that will work if you get an answer that the rule isn't saying.

Observer: What if we find one time that is true, then is it a rule?

Megan: Yeah.

Observer: If it is true one time?

Laura: No, just because it is true one time doesn't mean it is a rule. If you do it a few times, if a lot of people do it a few times you can make it a rule, and if somebody gets the answer that doesn't follows the rule… then it wouldn't be a rule.

Observer: Then you just toss the whole thing out?

Laura: Yes.

In this example, Laura and perhaps others show an understanding that although a counterexample will make a conjecture false, one example does not prove a conjecture to be true. Most children seemed to believe, however, that lots of examples would prove a conjecture to be true. Often when Ms. Keith asked the class if a conjecture was true, children responded by offering examples for which the conjecture worked. When she asked the children if the conjecture would work for every number, they said that it would. However, their only way to explain why a conjecture would always be true was to offer examples.

By the end of the year, some of the children became more sophisticated in thinking about justification. For some of the conjectures that were easier to justify, children offered general explanations that did not rely on examples. In the following segment which took place on April 1, they talked about why they knew that the conjecture "If you subtract zero from a number you will end up with the same number" was always true.

Ms. Keith: How do you know that that will always be true?

Mitch: Well, you could put 12 minus 0 is 12, 2800 minus 0 is 2800, it will always work.

Ms.. Keith: Ollie, how do you know that that will always be true?

Ollie: Well, because zero is like a nothing, so when you are minusing, minusing is taking away, so you are taking away nothing, so it just equals the same thing you started with.

Ms. Keith: Ok, does anyone else have another way of saying that? Kate?

Kate: Because … you minus the zero from the number, zero is acting like zero…

Kathy: I know it is going to be true because like Kate, something you take zero away from wouldn't change.

Ms. Keith: Carl?

Carl: Like 11 minus 0 equals 11.

Ms. Keith: Will that always work?

Laura: Well, if I had 11 things… [Counts out 11 cubes] and I take zero away, I still got 11. It looks like this for every number.

Megan: I think it will always work, I tried it and tried it and tried it.

As this example shows, some children were able to think in general terms. about how they knew this conjecture was always true. Ollie offered a principled explanation. Although Laura's explanation was tied to an example, it did include some principled understanding of subtracting zero. Although Kate and Kathy thought they were offering justification, they were only restating the conjecture. Several of the children offered examples, and Megan thought that trying it a lot of times had convinced her that it worked. Although Ms. Keith frequently asked the children how they knew that a conjecture was always true, individually and as a class they never got to the place where they valued principled explanations over examples.

## REPRESENTING CONJECTURES WITH NUMBER SENTENCES

The third theme related to the development of algebraic thinking involved the use of open number sentences to represent conjectures. Most of the arithmetic exercises in the class were posed in the form of word problems. Children often wrote number sentences after solving story problems to reflect how they solved a problem, using a ☐€to represent an unknown quantity in a story. In January, open number sentences and variables emerged as a way of expressing conjectures. The class was discussing whether "Zero plus a number equals that number" and "a number plus zero equals that number" were different conjectures.

Laura: They are different. The first one is like zero plus number equals number [she writes 0 + # = # on the chalkboard], and the other is like number plus zero equals number [she writes # + 0 = #].

Ms. Keith: Can anyone else write those conjectures with something other than a number sign?

[Mitch wrote * + 0 = * and 0 + * = *. Carla wrote $s + 0 = s$ and $0 + s = s$, and Jack wrote $m + 0 = m$ and $0 + m = m$.]

[….]

Ms. Keith: Jack just wrote zero plus m equals *m*. What were you using the *m* to mean?

Jack: I meant it to mean any number.

Ms. Keith: Then we could put any number in there? What if I put a 2 here and an 8 here?

Laura: No…you could put a 2 here and a 2 here. You have to put the same thing in both places.

There was further discussion of this idea. Ms. Keith told the class that mathematicians call these symbols variables and they use variables to stand for any number. She also told the class that Laura was correct, that when a variable was used more than once in a number sentence you had to use the same number for both variables. The next day this rule became known as the mathematician's rule for repeated variables and was written on a piece a paper and taped to the wall. After this discussion, Ms. Keith often asked children to write a conjecture they generated as an open number sentence. Children were able to do this for the conjectures dealing with zero and had some success with expressing even and odd numbers with variables.

## INTERVIEW RESULTS

The results of the May interview are summarized in Table 1. Thirteen of the 17 students interviewed correctly answered both interview items assessing understanding of the meaning of the equal sign, and only one student missed both items. By way of comparison, only one student in a comparable class in the same school correctly solved the equation: $8 + 3 = c + 2$.

Thirteen of the students remembered one or more conjectures from the class, and all 13 could represent conjectures using open number sentences. One additional student could not remember a conjecture, but could use a number sentence to represent one given to her by the interviewer. Six students came up with some form of the conjecture $a + b - b = a$, but only 4 of them used the conjecture to simplify the equation $26 + f + f - f = 30$. Nine students solved this equation using trial and error. Eight students correctly solved the equation $c + c + 3 = c + 8$, but 7 of them used trial and error, and only one recognized that the two Cs on opposite sides of the equal sign canceled each other out.

Number sentences were used regularly in the class to represent simple word problems involving a single arithmetic operation, and 14 students wrote a number sentence ($53 - s = 27$) to represent a word problem describing corresponding action.

Four students correctly wrote a single number sentence to represent a more complex situation that involved a repeated variable. Two additional students wrote number sentences for part of the problem, solved them, and then used the answer to write a second number sentence for the remainder of the problem ($24 - 4 = 20$ and $20 \div 2 = 10$). The students had not seen a problem like this during the class, and writing a single number sentence for the problem appeared to require more insight about the relations among quantities in the problem than most students were capable of.

## STUDY 2 CONCLUSIONS

The classroom case study results generally were consistent with the results of Study 1. Students in both studies readily made generalizations from specific cases illustrated by true-false or open number sentences. The generalizations that students most readily articulated involved zero: adding zero, subtracting zero, or subtracting a number from itself. A few students were capable of using these conjectures to generate additional conjectures like $a + b - b = a$.

TABLE 1
STUDENTS GIVING CORRECT RESPONSES TO SELECTED END-OF-YEAR INTERVIEW TASKS

| Task | Students Giving Correct Responses ($N = 17$) |
|---|---|
| Equality | |
| $a = 5 + 3$ | 15 |
| $8 + 3 = c + 2$ | 14 |
| Remembered one or more conjectures | 13 |
| Represented conjecture with number sentence: $t + 0 = t$, $h - h = 0$ | 14 |
| Solved $26 + f + f - f = 30$ | 13 (4 used conjectures, 9 used trial and error) |
| Solved $a + a + 3 = a + 8$ | 8 (1 used conjecture, 7 used trial and error) |
| Represented word problems with an open number sentence: | |
| Kevin had 53 stickers. He gave a bunch of stickers to his mom. Now he has 27 stickers. How many stickers did he give to his mom? $53 - s = 27$ | 14 |
| Ms. Keith bought a package of 24 stickers. She wants 4 stickers for herself and then she will let McKenzie and Mellery share the rest of the stickers. How many stickers will McKenzie and Mellery get? $24 = a + a + 4$; $24 - 4 = g + g$; $24 - 4 - ? - ? = 0$; $24 - m - m = 4$ | 4 (2 other students correctly used a single number sentence for part of this problem.) |

## *Conclusions*

Most of the students in Study 2 learned to deal with the equal sign as expressing a relation, although this concept was fragile and required that students regularly see nonstandard forms of equations with more than a single number after the equal sign. The results of the two studies suggest that the equal sign can be introduced and understood as expressing a relation as early as the first grade.

Many of the first- and second-grade students experienced difficulty in coming up with justifications that went beyond examples, although a number of students did recognize the necessity of more generalizable justifications. It appears that initially students thought that proof by example was a legitimate form of argument for establishing generalizations about all numbers. They recognized that a single case was not sufficient to justify a conjecture, but they thought that a number of students trying a number of cases was valid justification. It probably is the case that most of the justifications they had seen involved examples, a form of argument that is easy to describe. Nevertheless, some students in the study did see the need for more general forms of argument, and it does appear that these primary grade students recognized that a single example was sufficient to show that a conjecture was not true. Overall, it appears that students in the primary grades can engage in formulating, representing, and justifying conjectures, even though their justifications might not always be sufficient to validate all the conjectures they are capable of identifying.

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