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In This Issue:

- Geometry in the Primary Grades
- Review of NCRMSE Research

Reforming Geometry

Visual reasoning is central to mathematics. It is an integral part of mathematical and scientific inquiry. Geometry also has another aspect that it shares with the broader field of mathematics—its potential use for modeling situations. Because of this aspect, geometry should—those who seek a reformed school mathematics curriculum claim—be integrated into the mathematics courses children study throughout their K-12 years. The NCTM *Standards* (1989) and other reform documents (MSEB, 1989; MSEB, 1990) call for increased content in geometry and spatial reasoning across Grades K-4, 5-8, and 9-12.

Geometry, a course devoted to visually guided thought, is taught to students in the United States at the 10th grade level. Most elementary school curricula cover the descriptive aspects of geometry briefly. Students learn to identify and name conventional shapes in elementary and middle school grades. Their 10th-grade course is built around two-column proofs of theorems in Euclidean geometry. Current instructional practice isolates the content of geometry from the content of other mathematical areas, a practice that begins with algebra in 8th-or 9th-grade classrooms. A discussion of algebra is carried in the Winter 1993 issues of the *NCRMSE Research Review*. The isolation of geometric content to a single high school course and the emphasis within this course on proofs contrasts with international practices—which give geometry and spatial reasoning more central roles in K-12 curricula. In The Netherlands, for example, children begin in the primary grades with such realistic geometry tasks as sighting and projecting, locating and orienting, and drawing and measuring (de Moor, 1991).

Research Program

The National Center for Research in Mathematical Sciences Education (NCRMSE) is undertaking a research agenda that will support the reform of school mathematics. One of its seven working groups focuses on the Learning/Teaching of Geometry. The research of this group is designed to facilitate a transition to a more integrated and more comprehensive approach for the teaching of geometric content. The working group on the Learning/Teaching of Geometry was formed in late 1991; it plans to carry out five major activities:

1) Investigate the informal notions that students bring to their understanding of geometric concepts and to their understanding of geometry as a whole. Researchers will explore children's intuitions about such fundamental concepts as angle, shape, and area. They will extend this research to include children's intuitions about the corresponding attributes of solids and motion. Their findings will form the cornerstone of their efforts to improve teaching by making teachers more aware of their students' thought process in the area of geometry.

2) Investigate the ways in which visually-guided thought influences mathematical thinking about number and quantity. Instruction in rational numbers, for example, is often based on the assumption that children have a working understanding of area. Similarly, instruction in the interpretation of graphs of functions often assumes that students can recognize the relationships between magnitude and positions or between change and position.

3) Explore alternatives for envisioning environments for the study of geometry. The computer provides the primary tool, but computer-based exploratory environments such as the *Geometric Supposer* (Kaput, 1990) or Log (Battista & Clements, 1988; Lehrer, Randle, & Sancilio, 1989) can be used to encourage children's explorations of the foundations of traditional geometry. The software tools make it possible to explore geometry in new ways. In a similar vein, some computer-based cognitive tools use geometric concepts to clarify the nature of scientific and related mathematic concepts. Chemists, for example, build polygons and polyhedra to understand molecules. Some members of the working group will examine systems that link the simulation of physical motion to the tables, graphs, or functions.

4) Study how research on students' intuitions about geometry and computer-based tools for the learning of geometry can effectively be merged to improve the teaching of geometry. There is evidence that Cognitively Guided Instruction (CGI) is effective with primary-grade children. How can teachers be provided with new knowledge about student cognition in geometry as well as new instructional tools? Will CGI instructional effectiveness be enhanced when teachers are provided with new tools as well as new knowledge about student cognition in geometry?

5) Critically examine the current geometry curriculum with the goal of suggesting change as a result of the research findings.

Working Group Activities

The members of the working group meet to discuss research related to the reform of K-12 geometry twice each year. They also communicate vie an e-mail network. At their 1993 meetings they began to identify and integrate the multiple views about geometry teaching and learning. The meetings give group members an opportunity to evaluate current research on geometry and to plan collaborative activities that extend the thrusts of current research projects. While all group participants are conducting research in areas related to the learning/teaching of geometry, some have backgrounds in mathematics or mathematics education, and others in cognitive or educational psychology, curriculum development, or software design.

The five major program activities are carried out by Working Group chair Richard Lehrer and NCRMSE staff members or by other members of the working group. The working group members are located predominantly at higher education institutions. While several members pursue research activities related to geometry with minigrants funded by NCRMSE, others combine these grants with funding from other sources. This is especially true of the group members who are designing and developing technologies to enhance visualization activities.

Preliminary Reports

Some of the investigations of the informal notions that students bring to understanding geometric concepts and to their understanding of geometry as a whole have been completed. Chair Richard Lehrer and staff member Cathy Jacobson have prepared a preliminary report on their findings, which appears in this issue under the title Reform in the Primary Grades. In this article, they report on their work with primary-grade teachers, the teacher workshops, CGI Geometry curriculum, and their classroom observations on the ways in which geometry content was implemented in primary-grade classrooms. Brief summaries of the work of other members of the working group are contained in the next section of this article. Much of their work incorporates a computer-assisted technology component. Brief descriptions of their research activities that relate to geometry follow:

Daniel Lynn Watt is a senior curriculum developer and geometry team leader for the Elementary Mathematics Project at the Education Development Center in Newton, Massachusetts. He and other staff members are developing a technology-rich geometry curriculum for elementary students. The geometry component of the curriculum, *Math and More*, builds on four assumptions: 1) Students learn geometry by using is for constructive and creative purposes; 2) Students learn geometry effectively when it is connected to their everyday lives and to the cultural artifacts they see around them; 3) Students benefit from an in-depth exposure to a few geometric ideas and concepts; 4) Student learning will be supported effectively when both highly directive structured and constructive or open-ended software activities are available.

Several *Math and More* units for Grades 1 and 2 were completed in late 1993. The units use computers, computer software, and video as tools to support students in understanding and using geometric ideas.

Specialized software environments or microworlds were developed for each of the units. In the Grade 1 unit, Maps and Movement, students use software to move a button-drive turtle as they build elementary maps from rectilinear path pieces, such as straight lines, corners, and intersections, with 25 landmark icons. Initial activities require that students move a pointer around pre-developed maps. Students give and follow directions, compute and compare distances, and find different routes to the same location. They then build their own maps and pose problems for other students. Later software environment activities include predicting the results of following a set of instructions on a map, solving treasure hunts using directions for clues, and making a treasure map. The video related to the maps unit tells a story of two animals who roam a neighborhood looking for human friends, Jesse and Raquel, who have gone off to attend school.

In the Grade 2 unit, Geometry in Design, students use software to build geometric quilt designs. Users begin by building a core square and retain the square in a special area of the screen; they then use their core square design to build larger quilt design elements. While they complete the design, they learn about patterns and shapes, congruence, symmetry and asymmetry, and visualizing or predicting the effects of such transformations as flipping or rotating a design element. The video related to the unit shows the richness of the geometric design found worldwide in arts and crafts. It animates the process by which geometric transformations can be used to build complex patterns from simple elements, and provides examples of quilt designs for students to copy and modify.

Martha Wallace has been working with two of her colleagues, Richard Allen and Judith Cederberg at St. Olaf College in Northfield, Minnesota, to help secondary teachers use computer-assisted software. They hope to make the teaching and learning of geometry in secondary schools a more engaging and dynamic activity with their project activities. Inspired by the NCTM *Standards*, their objectives are 1) to help teachers develop the knowledge, skills, and confidence necessary in using computer-based tools to transform their geometry classrooms into a mathematical community where students explore, conjecture, verify, and communicate mathematically and teachers are their partners in inquiry rather than correct-answer authority figures; 2) to enable teachers to share their expertise with other pre-service and practicing teachers.

While their research is not yet complete, Wallace and colleagues note that 90 percent of their participants indicate increased confidence in knowledge of geometry, knowledge of pedagogy and computers related to geometry, ability to develop instructional materials that incorporate computer technology, and ability to mentor other teachers. A similar percentage believe that their students became more motivated when learning geometry in computer-assisted classrooms.

The researchers conclude that teachers new to their project require at least two years to integrate and implement their new geometry knowledge and teaching strategies. Trying new techniques with their traditional curriculum initially eased the transition to a revised curriculum for teachers. But equally important were extended school-year support services and networking opportunities that provided support when they were faced with change-resistant individuals, administrators, colleagues, or their students' parents.

Both the NCTM Standards and the addenda series book, *Geometry in the Middle Grades* (NCTM, 1992), recommend that the middle school geometry curriculum provide many opportunities for students to explore their environments. Exploring their own familiar environments should, according to members of the Cognition and Technology Group (CTG), located at Vanderbilt University in Nashville, Tennessee, increase students' perception of the importance of geometry in their worlds. Members of the CTG group surveyed middle school students and teachers to determine their views of geometry. The responses indicated that both groups had limited views of the uses of geometry.

To provide opportunities for teachers and students to see, understand, and use geometry in the world around them, the CTG (1992) created the Jasper Adventure Series which consists or problem solving videodisc adventure stories based on geometric content. Based on a theoretical framework that emphasizes the importance of anchoring or situating instruction in meaningful problem-solving contexts, the materials reflect the principles of NCTM *Standards*, particularly in building on the recommendation that activities emphasize complex, open-ended problem solving that connects mathematics to other subjects and to the world outside the classroom.

Two of the Jasper Adventure Series videodiscs highlight the nature of geometry in wayfinding and measurement. Each requires a minimum of three to five class periods to solve the problems they contain. They are designed with video-based extension problems that help students deepen and extend their knowledge of the mathematical concepts they used with the original videodisc problems. The extension problems are designed so that teachers can use them flexibly in order to meet the needs of particular students.

Teacher users of the two videodiscs participated in a two-day workshop before using the materials with their students. Surveys showed both teachers and students improved dramatically in their ability to identify uses of geometry. While teachers thought they could use the materials more effectively when they taught them again compared to their first year, they expressed a strong desire to meet at least annually to share strategies and information.

Michael Battista of Kent State University and Douglas Clements of the State University of New York at Buffalo are examining student's learning of two-and three-dimensional geometry. Their work is part of a larger project to develop and test K-5 instructional units. They believe that children represent space based on their actions, so they designed Logo turtle and other non-computer activities to help students investigate the notions of length and rotation in the context of paths that are records of movements.

Three student strategies for solving two-dimensional length problems in the path contexts were apparent: 1) Some did not partition lengths, nor did they relate the number for the measure with the length of the line segment; 2) Most drew hash marks, dots, or line segments to partition lengths—they appeared to need these means to quantify length; 3) A few did not use a partitioning approach, but did use a quantitative concept when discussing the problem and drew proportional figures. It appeared that the third group created an abstract of length or a conceptual ruler—not a static image but an interior process of moving visually or physically along an object and segmenting it—that they projected onto unsegmented objects.

The researchers have begun developing descriptions of the errors students make when determining the number of cubes in a three-dimensional array of cubes. They have found that even though students, beginning in the 3rd grade, are taught a procedure for finding the number of cubes in a three-dimensional rectangular array, fewer that 40 percent of 5th graders can conceptualize such an array to enumerate its cubes in a meaningful way. This work supports the view that mentally constructing a three-dimensional cube array is a complex process involving numerical and spatial structuring supported by coordination and integration operations. To determine the number of cubes in a three-dimensional array, students must coordinate orthogonal views of faces, then integrate these views to construct one coherent mental model of the entire array of the cubes—an extremely difficult process for elementary school students.

Kenneth Koedinger, Carnegie Mellon University, is developing software to provide high school students with technology-based opportunities to be creators rather than consumers of mathematics. The software will be developed to engage students in discovering and evaluating geometric conjectures aided by computer-based tools for performing experiments and writing proofs.

Employing the ACT theory of cognition, Koedinger and a project team are developing computer simulations of the reasoning processes involved in conjecture discovery and evaluation. The simulations or cognitive models serve as a basis for providing students with individualized computer tutoring. Interprocess communication technology will be used to add to this tutoring support to such existing software as the *Geometer's Sketchpad* (1993).

For further information on the Learning/Teaching of Geometry Working Group, readers may contact its chair, Dr. Richard Lehrer, at the National Center for Research in Mathematical Sciences Education, University of Wisconsin-Madison, 1025 W. Johnson Street, Madison, WI 53706.

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Geometry in the Primary Grades

By Richard Lehrer and Cathy Jacobson

Cognitively Guided Instruction (CGI) is based on the premise that primary-grade instruction in mathematics should begin with a teacher's understanding of how children think about mathematics. In the research reported in this article, the CGI principles have been used by the authors to develop collaboratively—through an ongoing dialogue with primary level teachers—an extension of CGI that includes geometry and spatial-visualization components.

CGI classrooms emphasize problem-solving tasks and communication between teacher and children and among children. CGI teachers create learning environments in which children actively construct their own knowledge. This contrasts with classrooms in which children are often passive recipients of knowledge from textbooks or teachers. The goal of CGI teachers is student learning that is based on understanding rather than on the memorization of mathematical rituals. For additional information on the CGI program, see the Spring 1992 issue of *NCRMSE Research Review*.

Knowledge About Children's Thinking

Teachers' knowledge about children's thinking about the area of number has been the key to the CGI program. A significant amount of research on how children's thinking about number develops has been completed. That is not true for geometry, particularly for primary school-age children. This three-year longitudinal study of the development of children's thinking about shape; measurement which includes length, angle, and area; depiction which includes drawings and graphing; and visualization had been undertaken to expand knowledge about children's thinking in the areas that compromise geometry. It began in 1992-1993 with 1st-, 2nd-, and 3rd-grade students, and the evolution of their conceptions about geometry over a two-year period—e.g., as the original 1st-grade students progressed to 3rd-grade, 2nd-grade to 4th, and 3rd grade to 5th.

This study is finding that young children's reasoning about shape and dimension is much more elaborate than some previous research has indicated. Earlier research, for example, suggested that children's ideas about shape were limited. Children, it suggested, considered only the overall appearance of a shape and its global similarity to other shapes. This study is finding that children use many attributes of shape to make comparisons. The children in this study have ideas about both two-and three-dimensional shapes that are rooted in qualities that have practical implications for the development of activities—e.g., a shape's orientation or its size. They rarely attend to conventional attributes or properties of shapes, perhaps because the conventions are useful for more formal geometry but have few implications for children's everyday use of geometric concepts.

While the children in the present study possess knowledge about is not yet organized into a system. In accordance with previous research, these children rarely articulate relationships among attributes of shape and, as a result, they seldom organize relationships into systems. They use an extensive vocabulary and show an awareness of attributes of shape in their descriptions of shape and pattern during the primary school years. Children, when asked to compare shapes, often talk about the acuteness of some of the angles using words such as "pointy," or the orientation of line segments using words such as "slanty," or the number of sides or faces. They also talk about what the shape resembles and other aspects of appearance. These research findings have been synthesized into a new model of learning informal geometry that blends cognition and perception.

The CGI Geometry Curriculum

These findings suggest that primary-grade children could benefit from a wider range of experiences in the area of geometry. To provide children with these experiences, an experimental curriculum was developed. Building on children's knowledge of the world and the types of knowledge they develop in the course of everyday activity, it contains the four major strands: (1) wayfinding, (2) depiction, (3) two-and three-dimensional shapes, and (4) measurement.

The Wayfinding strand grows from children's learning about space by walking or moving in that space. Children, for example, find their way to a friend's apartment of house and as they do they develop knowledge about position and direction during preschool years. Wayfinding is connected to shape as well. The trace of the path taken while finding one's way can be thought of as a shape.

The Depiction strand grows from children's rich history of drawing before they enter school. Children's knowledge of depiction is extended in this curriculum as they develop drawings in different views of objects, integrate other children's drawings of views as they construct an object, or use or invent other graphical conventions to describe space such as maps and nets.

The Two- and Three-Dimensional Shapes strand grows out of children's play with blocks and other building materials. Rather than simple identification of shapes, the CGI Geometry curriculum emphasizes the uses of shape. Children build shapes from different nets, design quilts and gardens, and investigate transformations as physical motions during the process of design. These activities incorporate the use of such computer tools as Logo.

The Measurement strand grows out of but also unifies the first three curricular strands. It also makes connections between the geometry and number strands of the curriculum. The wayfinding component of the curriculum includes problems of length, the distance between two locations, and angle measurement, as well as the direction one must take to find a location. Students often find that their solutions to measurement problems in wayfinding apply to other problems involving shape. They describe similarities among shapes using measures of length or angle, or they use depiction to determine the appropriate scale for a map. Measurement problems such as finding the area of the floor of a classroom anchor descriptions of space to such number topics as multiplication and place value. As an aside, some historians think geometry initially was developed by those who surveyed or measured certain parts of the earth.

Teachers and CGI Geometry—Professional Development

The primary-grade teachers who participated in this research were provided with a series of workshops that took place throughout the school year. The series—approximately 60 hours—was led by a CGI mentor teacher and a project researcher. Workshop participants solved problems similar to those designed for the students in their classrooms. The workshops emphasized the evolution of children's thinking about the conceptual issues that were raised by the problems. Often the geometry problems were novel and provided teachers with increased content knowledge. Since teachers taught in different buildings, an electronic network was developed to facilitate communication between teachers and with the CGI mentor teacher and researchers.

After the series of workshops and a year of CGI work with their mathematics classes had been completed, teachers—with project staff—began to create the CGI Geometry curriculum. They began to develop problems and tasks that would provide effective windows through which to view student thinking. Teachers reflected on how to get students to reveal their thinking about geometry while creating the curriculum. Such reflection served as a catalyst for refining or redefining teachers' views on what is important in a geometry curriculum for primary school students.

Changes in teachers' beliefs and practices

The changes in collaborating teachers beliefs and practices with regard to geometry were examined in several ways. The teachers participated in an interview at the beginning and end of the school year and teaching practices of selected teachers were observed three times a week throughout the school year. The observations permitted researchers to relate a teacher's perceptions of changing beliefs with changes in the same teacher's classroom practices. Over the course of the research, these data revealed dramatic changes in how teachers thought about geometry and in their teaching practices.

Before they began the series of workshops, teachers saw geometry and spatial reasoning as only a minor adjunct to number sense and viewed their primary role as one of helping children develop number sense. Their idea of an appropriate geometry task reflected traditional textbook exercises. After the workshops and classroom experience with the CGI Geometry curriculum, teachers had broader conceptions of the domain of geometry and gave student thinking a larger role in curriculum planning. Teachers also said conceptual tools that parallel those available for classroom work with number—e.g. Logo or polyhedrons—would help them promote the development of spatial reasoning in primary-grade students.

Teachers talked about students' range of understandings of space—discerned by teachers as children completed activities—as well as the importance of conversation and discussion to students in their development of spatial thought. Student thinking became a more prominent concern of teachers, as indicated by one who said, "For the student who struggles to explain his strategy with number problems, we have to remember to design geometry problems so that the same students can also find a way to demonstrate the strategy he used with geometry problems."

The observations of teachers' classroom practices suggested that changes in beliefs about the teaching and learning of geometry were echoed in three major ways in classroom practices. First, the number of problems involving geometry and spatial visualization increased noticeably during the school year. In addition to problems involving shape—those most familiar to teachers—teachers began to pose problems in wayfinding, depiction, and measurement that reflected the major components of the CGI Geometry curriculum.

The character of the problems posed by teachers within the four strands changed. At first, teachers posed problems about shape that were variations on recognition and identification activities, but over time the problems on shape engaged children in constructing shapes and using the shape in design. Rather than simply asking students to identify squares and rectangles, students were asked to create compositions of triangles using "core squares," and then to transform the core squares using slides, flips, or turns on their compositions to create a quilt design. They selected some of the problems developed by Dan Watt, a member of the NCRMSE working group, The

Learning/Teaching of Geometry, whose work is described in a previous section of this newsletter.

Teachers' views on the learning potential associated with problems in geometry changed dramatically. At first, teachers posed problems and children solved the problems. There was little discussion of students' solution strategies and few connections were made between the activities students performed and their thinking about them. During the first few months, the teaching and learning of geometry was activity-based and emphasized a product. Teachers were more adept at focusing on process for number problems than they were with geometry problems. They were more familiar with the number curriculum, and they knew more about student thinking about addition and subtraction than about geometry. Over time, they began to focus on student thinking in geometry, and student conversations about different ways to solve the same geometry problems occurred with increasing frequency. The conversations emphasized similarities and differences among different solution strategies and highlighted the need for students to justify—verbally or with drawings—the spatial actions they were taking.

As the year progressed, teachers and students began to make connections among depiction, pattern, position, and measurement, as well as among these components and number components. While carrying out a wayfinding activity, children measured length and angle, depicted paths and coordinated their paths using a to-scale map, modeled their actions in a small-scale space using Logo, and described the shapes made by different types of paths that started and ended at the same point. Teachers used many opportunities to create bridges from other curricula to geometry. During a unit on the geography of islands, a teacher was observing posing problems involving depiction—e.g., draw and island; and measurement—e.g., order the drawings according to area. During this task, children talked about different strategies for counting fractional pieces and suggested possibilities for units to measure area.

Teachers also used their growing understanding of children's thinking about geometry to make instructional decisions. This informed their decisions to allow further discussion or to pose other types of problems. Classroom teachers began to see themselves as researchers who could document student thinking about geometry and design activities that would make students' thinking more apparent.

An example of the use of understanding occurred when a teacher posed a problem about drawing-to-scale. Students were presented with an adult hippo drawn inside a coordinate grid and were asked to draw a baby hippo inside a scaled-down grid. As the teacher watched their progress, she noted several strategies that ranged from artistic attempts that did not follow any of the grid scaffolding, to semi-grid work in which only tails and feet were coordinated with the grid squares, to free-hand drawing. Only one student used the coordinate structure to draw the baby hippo. The teacher had some of the students share their work and then developed a class discussion about how best to draw the baby hippo so that it looked like a smaller version of the adult.

Thus the teacher obtained data about how each of her students approached the problem. Based on this data, she designed a set of activities in which the scale of each of the axes differed. In one instance, the scale was "shrunk" for only one of the axes, which caused students to draw "flat" hippos. She provided grids of different sizes, with axes scaled with the same multiple, for students to use in creating hippos of different sizes. As students experimented with the different scalings, they talked with other students and the teacher, providing the teacher with opportunities to observe their growth in understanding. The teacher revisited the topic later in the year in other contexts and continued to explore the growth of students' understanding of scale.

Students and CGI Geometry

The performances of children participating in CGI Geometry classes were compared with those participating in a longitudinal study of CGI in the primary grades. For additional information on the CGI longitudinal study see the Spring 1992 issue of NCRMSE Research Review. Comparisons were also made between 2^{nd}-grade students' performances on a variety of geometry and number problems at the beginning and end of the school year.

The children participating in CGI Geometry did as well solving problems in number as their counterparts in the CGI longitudinal study. In addition, participating children, at the end of the one-year period, showed large differences in conceptions of geometry. They were able to think of shapes as paths as well as static figures, and they were much better than their counterparts in thinking about the relationship among properties of shapes. Their understanding of measurement also showed growth; they were able to reason about appropriate units of measure in the context of problems about length, angle, and area, and they could solve area problems involving irregular as well as regular figures. Marked improvements in their solving of wayfinding and depiction problems were also noted. Most were able to draw the top, side, and bottom views of a solid, and to integrate top, side, and bottom views in order to construct a solid.

Participating students' involvement in and effort with geometry problems reflected their uniformly high level of involvement with mathematics in their classrooms. Concept maps of their ideas about the content and nature of mathematics were developed; the maps revealed that geometry and tasks involving visualization were increasingly perceived as part of mathematics, rather than as a separate subject, such as art. And, as with number problems, they saw geometry problems as having more than one method of solution.

Future research will continue to examine changes in teaching practices and changes in student conceptions of geometry. The same teachers who participated in the first year of the study are now beginning a second year using the CGI Geometry Curriculum in their classrooms. The initial classroom observations made by researchers this year suggest that teachers with a year of experience with CGI Geometry are more effective in guiding the geometry activities of their students and that the student change observed during the second year is likely to be greater than that observed during the first year.

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Review of NCRMSE Research

The authors of a previous article in this newsletter stated that Cognitively Guided Instruction (CGI) is based on the premise that instruction in mathematics should begin with a teacher's understanding of a how children think about mathematics. This article takes excerpts from a guide for primary-grade teachers by Richard Lehrer, Elizabeth Fennema, Thomas Carpenter, and Ellen Ansell. The guide, *Conitively Guided Instruction in Geometry*, was designed to provide collaborating teachers with introductory information on children's thinking about geometry. It was developed using the preliminary findings obtained in ongoing research. The guide—about the informal notions children bring to the understanding of geometric concepts and how these concepts develop—will be refined as additional data become available.

Recognizing Shapes Through Patterns

Patterns are an important part of geometry. The key to the formation of any geometrical pattern is the repetition of some unit or form. Artists make extensive use of repeated elements to create pleasing patterns. Quilts and rugs, for example, are often formed by creating a core design and then repeating this design. Similarly, if we look at nature, honeycombs form a recognizable pattern because a three-dimensional cell is repeated in all directions at once.

Although it may not seem obvious at first, shapes such as squares, triangles, and prisms are patterns too. A square is a pattern because it involves several kinds of repetition: The sides are congruent and are of equal measure; so, too, are the angles; and there are several lines of symmetry—e.g., folding along the diagonal results in two congruent pieces that, when superimposed, fit exactly.

Reasoning About Shape

Just as children develop a number of strategies for solving arithmetic problems, children also develop a number of ways of reasoning about patterns that make up shapes. As children's reasoning about patterns and form develop, their understanding of the nature of shape and pattern changes. There are three major ways that children reason about shape and pattern: resemblance, attributes, and properties.

Resemblance—Direct resemblance

The most direct form of reasoning about shape is to identify a shape or pattern by its resemblance to previous shapes or patterns. Children often say things like, "it looks like a ramp," "it's a squished-in rocket," or "it's a bit squarish." Here children are relating an unknown shape to one that is, given their experience, more familiar. A square, for example, is recognized as a type of box or as something that looks like the squares depicted in books. They relate a particular shape to typical examples of shapes that they know. When children are asked to find instances of shapes in their environment, they may use this form of reasoning to provide many examples.

Reasoning that relies on resemblance to visual prototypes makes the orientation and size of shapes especially salient. For instance, children often suggest that changing the orientation of a shape makes it a different shape. The two squares depicted in Figure 1 are often referred to by primary age children as a square and a diamond or "tippy" square. In a similar manner, some triangles may not be viewed as triangles because they have two sides that are longer in relation to the third and are not prototypical in children's experience. Two shapes may be viewed as belonging together because of their "skinniness" or "fatness." Skinny generally means that the shape's proportions are different from those of the prototypical figure. A child could conclude that the rectangle and triangle displayed in Figure 2 are more alike, by virtue of their being "skinny," than the two triangles displayed.

Indirect Resemblance

Mentally changing a shape so that it becomes one that is known is another form of resemblance-based reasoning. It is, however, an indirect form of resemblance because they student transforms an image and then compares it to her/his prototype. In this process, rectangles are described as "pushed-out squares," parallelograms as "bent rectangles," and pyramids are mentally shaved so that they come out to resemble cones.

Resemblance-based thinking about shape is varied. Chiefly it relies on visual comparisons in a manner similar to children's first strategies for arithmetic problems, which rely on direct models of experiences. This is not surprising. Adults too will try to classify—by analogy or resemblance—something that is seen for the first time.

Attributes

At a second level of complexity, children reason about the attributes or characteristics of shapes. Attributes are elements that are noticeable and explicitly represented by a child: corners, edges, lines, and others. Children's perceptions of shape attributes are often very different than those of adults. Their descriptions of the attributes of shapes include words such as "pointy" for angles that are less than 60 degrees, "slanty" for line segments or faces on solids that are not oriented either vertically or horizontally, "points" for the vertices of shapes, and "edges" for the sides of plane figures.

When children first start to reason about attributes, they do so in a relatively uncoordinated way. The number of edges or sides does not necessarily have a relationship to the number of points or angles, for example. Seeing a six-sided shape, children often notice that there are six sides or “edges,” but must count the number of “corners” (angles) to determine their number. At the next stage of reasoning, children develop coordination among attributes of a shape so that if a closed shape has four sides, then it has four angles. Counting the numbers of sides and then the number of angles and talking about them plays a key role in children’s development of this stage of reasoning.

As children coordinated attributes of shapes, they also begin to recognize that some attributes are unaffected by other attributes. Rotating a triangle, for example, does not affect the number of sides or corners; large triangles have the same number of sides as small triangles and skinny triangles have the same number of corners or angles as fat triangles.

Properties

Children’s thinking about attributes leads to the development of reasoning that views shape as a collection of properties—e.g. squares have four sides; the opposite sides are parallel, and there are four right angles. These properties of shape form an integrated system that defines a figure; taking one of the properties away changes the shape. In contrast, attributes are not yet organized into a system so that the critical attributes that define a shape are distinguished from the non-defining or the less critical attributes. Children’s efforts to coordinate attributes leads them to develop properties. Measurement also plays a key role in students’ development of property-based reasoning. If students can measure angles, then they have a basis for recognizing the definitional role of angles for many shapes. Right triangles are distinguished from other triangles, for example, because they possess one right angle.

Property-based reasoning is the key to students’ increasing the ways they identify and think about shape and pattern. They learn that when the collection of properties changes, so does the shape: If a four-sided shape has one pair of parallel lines, it is a trapezoid; if it has two pairs of parallel lines, it is a parallelogram; if a four-sided figure has two pairs of parallel lines and at least one right angle, it is a rectangle and others. The collections of properties come to include increasingly abstract ways of looking at patterns, including lines of symmetry as well as differentiation of reflection and rotation symmetries, and scale.

Children eventually establish systems of relationships among the properties of figures. For example, if the opposite sides of a quadrilateral are congruent, then so are the opposite angles. These relationships among properties help children establish logical relationships among figures. A square becomes a kind of rectangle because it has all of the properties of a rectangle, even though it does not resemble a prototypical rectangle.

Assessing Reasoning About Shape

Conversations are an important way to find out how children are thinking about shape. Here we focus on a few simple problems that serve as conversational aids. One type of problem has to do with shape classification. Children are asked to find all the triangles or rectangles or octagons or cylinders that they can in a collection of shapes. Children's justifications provide windows to their thinking, and their conversations generate new ideas. Shape classification can generate conversation and windows to thinking that go well beyond mere identification. In addition, for example, to asking children to decide if a shape is or is not a triangle, ask a question about a movement, "If I pull on this corner here, will it still be a triangle? Why?" Other conversational questions include "How did you decide that was a triangle?" "Why are these two figures both triangles?" "What goes with this (present a new shape)?"

Table 1—Questions for a Triangle Classification Task

Problem Type	Example Question
Identification	What is this (number 1*) called?
Description	What makes it a triangle?
Comparison	How is this one (number 1) like this one (number 3)? Different?
Classification	Does (number 1) go with this triangle (number 4)?
Generation	How is this triangle (number 1) made?
Transformation	If I pull here (point to the top vertex) will it still be a triangle?
Conjecture	Someone put this one (number 4) with this one (number 1). Why do you think they did that?
Justification	Do you think they were right to put this one (4) with this one (1)? How can you be sure? Can you convince someone else? What makes a good argument?
Problem Posing	Can you make up some new problem about shapes that you would like to try out?

* *Task inspired by Burger & Shaughnessy (1986)*

Classification tasks can be used as windows to children's thinking. Table 1 displays a set of questions that can be used by teachers as probes to get more information about children's thinking. Each question generates a slightly different problem, with later questions that elicit an attribute-based or property-based reasoning. The following are excerpts from a conversation between a teacher and her 1st-grade children during a triangle classification task. Sixteen children in a 1st-grade classroom were seated on a rug. The teacher sat at the side of an overhead projector. Children faced a screen pulled over the blackboard. The teacher projected an overhead transparency with nine shapes (see Figure 3) onto the screen and asked children for their ideas with the words, "Put a T on every triangle."

Julia: (points to isosceles upright) Well, if it has one, two, three, corners, and three sides then it would be a triangle. (Julia reasons about the properties of number-of sides and number-of angles as defining a triangle.)

Teacher: Okay, this one (pointing) has a flat line on the bottom.

Ellen: Yah. Almost like this one except this one is on its side.

Teacher: Oh, so you're saying, what about if I do that (turns around so base is parallel to ground) to it? Now is that one on it's side still? (Teacher rotates the figure.)

Ellen: Well, that depends because it can have different looks even though it was on its dies then. When it was on its side, it looked like it was going like this, but now it looks like it's straight.

Teacher: Oh, okay.

Ellen: Even though you didn't fully turn it. (Ellen reasons about the attribute of orientation and decides that it's not important here.)

Azim: It's half of a triangle.

Teacher: Half, why is it half of a triangle?

Azim: Cause the bottom is missing. (Azim compares the instance to a visual prototype. This is an example of resemblance-based reasoning.)

Ellen: I think it's a triangle because, well you know how there's like this tower of London or something? (Ellen reasons about real-world prototypes, and example of reasoning by resemblance.)

Teacher: Ahuh.

Ellen: That's like a triangle and that looks a lot like it, except it's not the same.

Teacher: What do you mean it looks a lot like it but it's not the same?

Ellen: I think it's not the same as the tower of London.

Teacher: But it looks similar to something you know then, that you've seen?

Ellen: Yah. It's a triangle that's really stretched out that maybe if you put it into one of the pictures with the T's on it, if you went like this with it (pushes top down), then it would be a regular triangle again and if you stretched it out it would be like that again. (Ellen reasons about a movement that would preserve resemblance to a prototypical triangle.)

Teacher: Oh, so which one do you think is a regular triangle? You said it would be like a regular triangle, what do you mean by a regular triangle?

Ellen: A regular triangle is one that's not really stretched out. (Ellen compares that proportions of the example to the proportions of the prototype—the "regular" triangle. This is an example of resemblance-based reasoning.)

Teacher: Do you see a regular triangle up there?

Ellen: Yes.

Teacher: Which one?

Ellen: (Ellen points to the triangle with the number one on it.)

Teacher: Oh, so to you, when you're thinking about regular triangles, you're thinking about that kind of shape.

Ellen: Yes.

Peter: Well, a needle (cone shape) can be a triangle if you cut the like round thing off; then it looks like a triangle. (Resemblance—movement and comparison to a visual prototype.)

Helen: (Goes up to screen with pointer) Even though the sides are stretched out, I think it's still a triangle because here's (point out three corners) one corner and here's another corner, and here's a third corner, and this (other triangle) has one corner, two corners, three corners. This has one, two, three sides, and this has three sides. (Helen reasons that the property of the number of sides and number of angles is important to classification, not the measure of the sides. This is a key step to seeing a shape as a collection of critical properties.)

Teacher: So you're thinking it doesn't matter what size the sides are. You're thinking it just has to have three? You're thinking it doesn't matter if it's two long ones and a short one, or if they are all kind of the same?

Ellen: I think it might be an ancestor.

Teacher: It might be an ancestor, what do you mean by that?

Ellen: I think it might have been like from the olden days. That's how they used to draw triangles. And they got smaller and wider and smaller and wider. (Ellen likens triangles to biological evolution, an unusual form of resemblance-based reasoning!)

Summary

The excerpts reprinted here are part of a 76-page guide prepared for teachers working with researchers investigating the development of a primary-grade curriculum CGI Geometry. Most of the article was taken from the chapter, Shape Through Pattern. Other chapters in the guide are Depiction, Thinking about Depiction, Direction, and Measurement, as well as an Appendix that discusses using Logo and a path perspective for shape, problem solving contexts, fundamental elements of dimension and shape, and a Glossary of geometric terms.

Reference

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