NCRMSE Research Review
from the National Center for Research in Mathematical Sciences Education

Volume 2, Number 1
Winter 1993

In This Issue:
-The Teachers and Algebra Project
-Review of NCRMSE Research
-An Annotated Bibliography of Publications

An Algebra for All Students

Reformers of school mathematics would teach a substantial body of mathematics to all of this nation's students. One course, algebra, however, creates a major barrier to reform efforts. Students find algebra one of the most alienating parts of their school curriculum. For a large proportion of 9th-grade students, algebra becomes a filter. Based on whether they are permitted to enter this course or their experiences in it, they come to view themselves as having little potential for involvement in further mathematical endeavor.

Most of the significant others in students' lives—parents, teachers, counselors, and peers—help to create and reinforce their limiting views about mathematics. American schools uniformly ask students to develop arithmetic skills before exposing them to algebraic concepts. A student's performance in algebra is the primary criterion used by parents and teachers to determine the fitness of 13- or 14-year-olds for a sequence of college preparatory mathematics courses. And the same criterion is used by counselors to route these students into—or out of—mathematics-related academic and career choices.

Other industrialized nations treat algebra differently, and they move larger numbers of students on to more advanced mathematics when compared to the United States. While educators here delay the study of algebraic concepts until secondary school, these concepts are systematically included as early as the 3rd grade in Japan (Fujii, 1992). Educators here treat algebra as an independent subject, isolated from the mathematics students have encountered in earlier grades and not integrated with that they will encounter in later grades. Some nations view algebra as a language; in that view, students require repeated exposure over an extended amount of time to become efficient in its use. In addition, algebraic concepts are taught in a manner that connects them with other mathematics content during elementary and secondary grades.

The National Center for Research in Mathematical Science Education's (NCRMSE) Working Group on Learning/Teaching of Algebra and Quantitative Analysis is reexamining the place of algebra in a core quantitative mathematics curriculum. According to the chair of the Working Group, James Kaput of the University of Massachusetts-Dartmouth, the nearly 80 group members include leaders of the major algebra-related research and development projects here and abroad, mathematics education researchers from this and other nations, doctoral students in mathematics education, and classroom teachers. Many members worked together earlier as part of a symposium on the graphical representation of functions or were participants in an interest group on algebra and technology.

Over the long term, the Working Group seeks to develop coherence in research activity and foster a common sense of "programme" regarding algebra research. It plans a series of research syntheses that describe commonalities among research findings and interrelationships among members' findings. Finally, it will develop a range of practical recommendations for teacher educators and for curriculum and tool developers.

Activities of the Working Group

There are three major interrelated activities of the Working Group. The total group is rethinking the school algebra experience in a fundamental way. Its members take the position that algebra is the study of only one kind of mathematical object—the function—and a small set of operations (both unary and binary) that can be performed on such objects.

Rethinking the Algebra Experience

Through its continuing discussions, group members are examining the character and purposes of algebraic reasoning, its relationship to other forms of mathematical reasoning, and other related topics. They are also discussing how the development of algebraic reasoning can be fostered in all students and how algebra could be translated across middle-school/secondary-school boundaries. Electronic networks are being used to insure that all members can be part of these group discussions.

Several group members met early in 1992 to plan for a series of research syntheses and empirical studies that would relate to the reform of algebra. While the series will be completed in a collaborative fashion, it was begun around six topics. They include and examination of the literature and a synthesis on the cognitive "shift" from arithmetic and quantitative thinking to formal, algebraic thinking by Jack Smith of Michigan State University; an analysis of the factors underlying the representation and symbolic manipulative power of new technologies by David McArthur of the Rand Corporation; a paper that clarifies the notion of symbol sense by Abraham Arcavi of the Weizmann Institute—Israel; a synthesis of the various conceptions and representations of function and an evaluation of their treatment in the literature by Alan Schoenfeld and his colleagues at the University of California—Berkeley; and a study of the evolution of the concept of function and the role the concept played in developing new forms of mathematics by Patrick Thompson of San Diego State University.

When the initial series of syntheses has been completed, several small group meetings or focus groups will be developed to address the series. These groups will include six to eight members and their deliberations will lead to the identification of the need for additional syntheses or empirical studies. Such topics as equity, tracking, assessment, and teacher development as these related to algebra, will receive increased attention from group members during 1993 and 1994.

The Teachers and Algebra Project

The Teachers and Algebra Project, a second Working Group activity, has developed the content of a Grade 6-12 curriculum based on the concept of function. It is gathering evidence about the teachability of the curriculum through extended clinical discussions and observations with teachers and their students. The project is directed by Judah Schwartz of the Harvard Educational Technology Center in Cambridge, Massachusetts. Director Schwartz is a principle investigator of the Working Group on the Teaching/Learning of Algebra and Quantitative Analysis. A longer article that describes the characteristics of the new curriculum and provides findings from a preliminary analysis of data obtained when project teachers began to use the curriculum appears in this issue of the *NCRMSE Research Review*.

Steps Toward a Revised Curriculum

Developing recommendations for curricula and tools is the third activity of this Working Group. Its efforts are framed by four fundamental assumptions about algebra that relate to quantitative relationship, functions as a central theme, new modes of representation, and algebraic thinking:

Algebra must be seen as part of a larger curriculum that involves creating, understanding, and applying quantitative relationships.

Algebra must now be seen as growing from quantitative reasoning at a fairly early age and extending to ideas and applications that have traditionally been viewed as the province of calculus. It must engage a wider set of analytical tools, especially graphic ones; and it must include connections to the wide variety of domains, both practical and theoretical, that use quantitative analysis.

The algebra curriculum should be organized around the concept of function.

Putting the idea of function, hence variable, at the center of the algebra curriculum has not been accomplished in the popular school algebra curriculum despite nearly a century of mathematically informed recommendations (MAA, 1923; NEA, 1969). Changed technological circumstances provide renewed urgency and opportunity for operationalizing these first two assumptions. The third and fourth assumptions reflect the new circumstances.

New modes of representation, graphical and otherwise, need to complement the traditional numerical and symbolic views of functions and the relations among them.

Historically, algebra has been identified with a set of formal propositions which evolved within static media to serve the scholarly interests of a small knowledge-producing elite. The new, dynamic, and highly flexible electronic media allow visual, graphical representations of quantitative relationships that are likely to be more learnable and applicable by the greatly enlarged segment of the population who must now learn and use them (Kaput, 1990). These representations include, but are not limited to, traditional coordinate graphs of functions.

Algebraic thinking, which embodies representation of patterns, deliberate generalization, and most importantly, active exploration and conjecture, must be reflected throughout the quantitative analysis curriculum.

The interactive capability of computers, coupled with their ability to perform routine computations, can greatly facilitate algebraic thinking particularly given the representational pluralism they make possible. Students can now test conjectures fluently, using different combinations of representations as appropriate, ranging from concrete, physically oriented systems (Greeno, 1989), to more traditional coordinate graphs (Schwartz & Yerushalmy, 1990), to standard symbolic formulas.

The framework for this project grows from a revised content analysis that draws explicit analogues with arithmetic. These analogues are shown in the box. Two strands will be woven together to develop the initial content for the course. One will deal with the formal mathematical structure, independent of its application to the modeling of the worlds of nature and people and their activities. The other will deal with the use of mathematical analysis as a language for modeling the students' world experience.

Moving important ideas in algebra and quantitative analysis to lower grade levels puts new demands on teachers. Many may not have had backgrounds that exposed them to this new content; nor did their education prepare them to teach it. Thus, whether all teachers can learn and teach the content of the new course must be studied.

According to Chair Kaput, while some pioneering work was undertaken earlier, the bulk of research on students and algebra was completed in the 1980's. The research findings indicate that students lack an understanding of the idea or variable, are unable to model quantitative situations, and have difficulties in parsing and operating on symbolic expressions (Booth, 1984; Clement, 1982; Kieran, 1983; Matz, 1980; Sleeman, 1984; Wagner, 1981). Later research showed that three fourths of 15-year-old students, when given a task, avoided algebra, while only 10 percent used algebra correctly (Lee & Wheeler, 1987, 1989). The shortcomings of current algebra curricula, according to Working Chair Kaput, seem to result from a disregard of the function concept. He identifies the five shortcomings as: a lack of clarity about mathematical objects involving an equal sign; not making graphical distinctions between equations, relations, and functions; presenting students with expressions rather than functions; providing students with too many variables in stressing the algebraic manipulation of symbols; and ignoring the differences between coefficient and variable.

While the goal of this Working Group is to rethink the way schools teach algebra, it has had to rethink the sequencing of algebra content, with a view of the changes that are possible when technology is utilized in classrooms. A list of the members of the Working Group and a list of references on the reform of algebra and related topics can be obtained from Working Group Chair James Kaput, Department of Mathematics, University of Massachusetts-Dartmouth, North Dartmouth, MA 02747.

References

Booth, L. (1984). *Algebra: Children's strategies and errors*. Windwor, Berkshire, England: NFER-Nelson.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education*, 12, 16-30.

Fujii, T. (1992). *Japanese students' understanding of school mathematics: Focusing on elementary algebra*. Unpublished manuscript.

Greeno, J. (1989). A perspective on thinking. *American Psychologist*, 44(2).

Kaput, J. (1990). *Reflections on the Supposer experience: What's particular and what generalizes?* A paper presented at the annual meeting of the American Educational Research Association, Boston, MA.

Kieran, C. (1983). Relationships between novices use of algebraic letters and their use of symmetric and asymmetric equation-solving procedures. In J. Bergeron & N. Herscovics (Eds.), *Proceedings of the 5th Annual Meeting of the PME-NA*, (Vol. 1, pp. 161-168). Montreal.

Lee, L, & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalization and justification* (Research Report). Montreal: Concordia University.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics*, 20, 41-54.

Mathematical Association of America (MAA). (1923). *National Committee on Mathematical Requirements of the MAA*. The reorganization of mathematics in secondary education. Washington, DC: Author.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior*, 3, 93-166.

National Education Association (NEA). (1969). *Report of the Committee of Fifteen on elementary education*. National Education Association, New York: American Book Company.

Schwartz, J. & Yerushalmy, M. (1990). *Seizing the opportunity to make algebra mathematically and pedagogically interesting*. Paper given at the NCTM Research Presession on the Graphical Representation of Functions, Salt Lake City, UT.

Sleeman, D. (1984). An attempt to understand students' understanding of basic algebra. *Cognitive Science*, 8, 387-412.

Wagner, S. (1981). Conservation of equation and function under transformation of variable. *Journal for Research in Mathematics Education*, 12, 108-118.

The Teachers and Algebra Project

By Judah L. Schwartz

This nation's students will need a substantively reformed mathematics curriculum if they are to face the challenges of the 21st century. The Teachers and Algebra Project is studying ways in which the traditional curriculum used in high school algebra can be reformed. Its three-year research plan builds on the beliefs that two factors are essential to the success of a reformed curriculum. Such a curriculum will need to contain coherent mathematical ideas, and teachers will need to learn the reformed content and find it acceptable. Thus the project began by reforming an algebra curriculum. It is now exploring teachers' reactions to the reformed curriculum, working with a small group of middle school and secondary teachers. This report describes the characteristics of the reformed curriculum in its first section. In the second section, a preliminary analysis of data obtained when project teachers began to use the curriculum is described. The Teachers and Algebra Project is directed by Judah Schwartz of the Harvard Educational Technology Center in Cambridge, Massachusetts, a principle investigator of the NCRMSE Working Group on the Teaching/Learning of Algebra and Quantitative Analysis.

The Reformed Curriculum

The mathematics of function acts as the unifying idea for the reformed curriculum. Two dimensions, mathematical objects and mathematical actions, provide it with structure. Mathematical objects such as numbers or functions are aspects of mathematics that are collectively understood and come to be viewed as "things" by those experienced in the knowledge and use of mathematics. This curriculum builds from the position that the function is mathematically and pedagogically the primary and fundamental objects of the subjects of algebra, trigonometry, probability, and statistics, pre-calculus, and calculus. Existing algebra curricula may confuse students because they confound equations and expressions, functions and relations, unary and binary operations, and variables and unknowns. The reformed curriculum clarifies these aspects by relating them to the central notion of function.

Key Mathematical Ideas

The "big ideas" of mathematics are not ordinarily apparent in early mathematics courses. They appear when students take more advanced courses or became committed to mathematics or a science that uses mathematics extensively. The big ideas included in the reformed curriculum are: representation, transformation, symmetry, invariance, scale, continuity, order and betweenness, boundedness, uniqueness, relaxation and constraint, successive approximation, and proof and plausibility. In this curriculum, computer graphics permit the introduction of important mathematical ideas early in the mathematical education of all students.

Functions

Functions traditionally have been represented in several ways (Harel & Kaput, 1991; Sfard, 1991): (a) the numerical-tabular, which can aid students in making the transition from arithmetic thinking to mathematical thinking; (b) the symbolic, which stresses the process nature of functions, and (c) the graphic, which stresses their integrity as entities as well as processes. In the traditional course in algebra, students may learn very little about the notion of function as an entity. This cripples students when they begin to learn calculus, which fundamentally is about the unary operations of differentiation and integration on functions as objects.

Certain properties and behaviors of functions appear more naturally and are more readily grasped by students through the use of one of the representations. Students comprehend the binary operation of composition more readily using an algebraic symbolic representation, but the unary operation of dilation/contraction more readily using a graphical representation.

Unary Operations

The design of the project's reformed curriculum began with the selection of a small but sufficient set of operations that can be applied to functions. Students seem to grasp unary operations (horizontal and vertical dilations and contractions, and reflections in the coordinate axes) rapidly if they encounter them first as graphical representations. It is also important to treat such operations in an appropriate modeling context using cognitively and pedagogically appropriate questions such as: Given the graph of a person's height as a function of age, what do each of the transformations shown on the board correspond to?

Binary Operations

Students seem to grasp binary operations readily if they encounter them first as symbolic representations. The binary operations on functions, which include the four arithmetic operations and composition, can be expressed both symbolically and graphically. Software environments that permit users, whether they be students or teachers, to manipulate functions in these ways have been designed and tested. A list of them can be obtained from the author.

Comparisons

Functions can be compared to one another; a binary comparison could involve R = s, R < S, or R > S. The use of an equal sign produces and equation or identity, while the use of a greater-than, or less-than sign produces an inequality. If functions are compared in a graphical environment, the solution set of the comparison is both evident and immediately accessible by identifying intervals where one functions lies on, above, or below the other. Such an environment, linked to symbolic representations, also provides a medium for the exploration of the manipulations that may be performed on an equation or inequality—for example, What operations on the functions being compared leave the solution set invariant? The use of computer graphics to support such inquiries leads quickly to a rather large set of syntactically complicated algebraic activities showing inequalities that can be reduced to a small set of graphical activities possessing easily recognized quantitative content.

Context

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) advocate activities that grow out of problems situations. Functions can be treated as mathematical objects—patterns worthy of interest in their own right—or as representations of patterns that exist in situations or phenomena. Regardless of the treatment, functions act to contextualize mathematics. At an elementary level, functions contextualize numerical computations and provide both a means and an end for the examination of quantitatively rich situations. Unlike the traditional word problems made up of a single set of numbers with a single numerical solution, functions draw students to explore situations for their regularity, to build insight rather than to compute single right answers.

Mathematical Actions
Transforming

All problems of the form, "simplify" "factor," "expand," and "collect similar terms," are instances of transformations of the symbolic form of a function and reexpress it in a different, but equivalent form. Transformed mathematical objects may represent some new aspect of the situation being modeled. If the average speed at which a car travels during a trip and the total time the trip takes are known, for example, new mathematical object can be generated by multiplying the two quantities to obtain a quantity that describes the distance covered by the car during the trip. The many types of transformations carried out on mathematical objects include arithmetic operations; various topological and metric changes; differentiation, integration, and composition of functions; and vector sums and products. Operations carried out on mathematical objects also come to be thought of as objects that permit the definition of higher order operations.

Modeling

Expressing, in mathematical form, the relationships between quantities in the real world is called modeling. In the reformed algebra curriculum, the mathematical expressions of these relationships are in the form of functions. Modeling allows the students to use his/her understanding of mathematical structures and their allowable transformations to reason about the situation being modeled. For a mathematical model to be useful, the mathematical elements and relationships must reflect those elements and relationships of the situation that the students regards as essential to what is being described, as well as the purpose for which it is being described.

Conjecturing

The search for and exploration of patterns in the interrelationships among mathematical objects can be called conjecturing. This aspect of mathematics has been sorely neglected by school mathematics, yet it lies at the heart of mathematical creativity. The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) suggest that students can begin to make and explore conjectures even before they can understand and use a wide variety of mathematical objects and the transformations that can be performed with them. This new curriculum conveys to students and teachers an implicit message that mathematics is not a closed and finished discipline, that they can go beyond learning and teaching the mathematics made by others to the making of new mathematics.

Teachers and The Reformed Curriculum

To insure that the content of the reformed algebra curriculum would be learned and accepted by teachers, the staff of the Teachers and Algebra Project began to work with core groups of three middle and three secondary teachers from suburban and urban schools in the Boston area. The teachers are volunteers who are motivated to change the way they teacher, but who have not been especially equipped or encouraged to change. Several dozen additional teachers have participated in week-long workshops and followup activities. On the basis of more than 1,000 hours of interview and observational data collected from these teachers, it is clear that their knowledge of the algebra domain has the same constraints as the curriculum they were educated to teach and the materials they presently work with in their classrooms. Some of these constraints are discussed in the lead article of this issue of the *NCRMSE Research Review*.

Five themes seemed to recur with regularity in an analysis of the data collected from project teachers during interviews, meetings, and workshops. These include (a) teachers as users of the verbal language of algebra, (b) teachers as learners of algebra, (c) teachers as users of graphical and symbolic representations, (d) teachers' rationale for teaching algebra, and (e) teachers as curriculum designers. The first three themes will be the focus of the next three sections. The teachers involved in this project did not possess as well-developed rationale for why students should learn algebra. While their understanding of algebra expanded dramatically, the process of applying that understanding to their classrooms was a slow and complicated one.

Teachers as Users of the Language of Algebra

Project teachers made noticeable changes in their use of mathematical language during their involvement with the project. They increased the precision with which they used mathematical terms and vocabulary. As they gained experience manipulating graphical representations of functions that paralleled their traditional manipulations of symbolic representations of functions, the mathematical language they employed included more visual metaphors.

During early meetings, teachers rarely used the term "function" and made few distinctions between such words as "equation" and "expression." After the meanings of these words were discussed, as self-consciousness emerged as teachers caught themselves using the word equation when they meant function. In later meetings, the mistakes became a source of humor, with careful listeners correcting their colleagues. Their confusion with the terms probably represented their confusion with the role of x when used as a variable or when used as an unknown.

The meanings of the words used in teaching algebra and the meanings of the words in relation to other topics or ideas were explored systematically in project discussions. Teachers considered whether they used certain words in ways that inherently lead to confusion rather than clarity for their students, and whether there is a logic regarding word meaning that could enhance rather than undermine teaching. As teachers and project staff increased their understanding of the mathematics that lies behind mathematical terms, they concluded that the traditional algebra curriculum does not possess a coherent language for its concepts.

Teachers were encouraged to become familiar and comfortable with software environments during project meetings. The software environments allowed teachers to manipulate functions in graphic and symbolic ways. Teachers began to use a greater number of words and phrases about seeing and moving: "Now I can see it." "If you get far enough away it looks like a parabola." "I can just compare the pictures." "Where did that hump come from?" "It's a quadratic, but I see, it is really flat." "We have to change the fatness factor." "Parabolas are 'fat and thin'." "Cubics 'float up here over the axis'."

Words about motion and action also became commonplace at project meetings. As teachers began to manipulate and modify functions graphically as well as symbolically, they described their actions verbally, "Let's drag this one over here." "All we have to do is stretch it and then push it up." Distinctions between functions that seemed compelling when represented symbolically could be seen in a quite different way when represented graphically.

Teachers commented that it is difficult to pull themselves away from the use of symbolic representations. They are now looking at the graphical representations of the constructs for which they used only symbolic representations in their classes. If they are assured that they will not have to forego the language of symbols in favor of the language of graphs, teachers admit that they feel empowered by graphical possibilities. While this use of language indicates movement, the extent and depth of this movement varies from teacher to teacher.

Teachers as Learners of Algebra

Teachers moved from their roles as "knowers" into those of questioners and learners during the series of project meetings. They began with an understanding of algebra that was "locally firm and globally fragmented and incoherent," in the words of one project observer. They had difficulty talking about the value of algebra outside of their classrooms. Teachers were taught new ways of thinking about and organizing the subject and provided with software that illustrated these new ways. Project researchers saw dramatics changes in teachers as they explored and discovered algebra from these perspectives.

When presented with a series of algebra problems in an early session, teachers began working on them with a firm belief and a sense of security in their knowledge of the subject. Most of them skipped introductory exercises or problems and went straight to those that were the most challenging. They soon had to retreat and rethink their approaches, however, as they realized that their understanding of the problems and concepts was based solely on a symbolic representation perspective. Teachers also spend early meetings exploring new software and the options provided in each of the packages.

The teachers' learning was, at first, a private matter. Their voices were low as they worked to solve problems, asking quiet questions of their partners or the project staff. They rarely commented about learning something new, seeing a new dimension of algebra, or understanding something that they had never before comprehended. Some would say, "How could a student infer that?" or "Maybe a really savvy student would get it, but I'm not sure about my classes." These remarks indicated that the teachers were learning new things; their conjectures about whether the materials could be learned by their students may, in fact, have been a projection of their own uncertainty about the material. There was a growing openness as group discussions evolved from individual questions. By the second session, teachers had become uninhibited, making remarks or asking questions of anyone near them: "Well, this certainly isn't my bag of tricks!." "I never had this understanding of algebra." " I never saw it this way until now." " I learned something new today.' "I don't have any answer for that and that is why this is new and exciting territory for me."

By the fourth meeting, teachers worked with an escalating excitement, delight, and rigor. Many telling remarks were made during a discussion about imaginary roots in the complex plane. Participants in this conversation stretched very hard to visualize a notion that had only been familiar to them in symbolic form. Their remarks included: "Until today I didn't have a model to deal with complex numbers." "Her we all are, trying to see what we really only know symbolically. This is a real struggle." "I have never had this visually." "Now I know why the complex numbers are in pairs." "How do I picture complex zeros? I don't!" "Are we in 3D?" "This is very exciting. That is a hard visual to get." "So that's where the imaginary numbers have been! Wow! I see. I mean imaginary is imaginary, so I never knew I could see them."

Teachers' Use of Graphical and Symbolic Representation

Teachers identified an interesting tension early in the series of project meetings. They noticed that they had developed a "symbolic fluency," or comfort with and reliance upon symbolic representation, early in their mathematics education. Most said they were still thinking in terms of numbers and symbols when they looked at graphical representations of functions. Project staff wanted to know whether teachers held a symbolic bias that could affect their learning and teaching of algebra, and when the teachers' symbolic and graphical approaches would become interwoven. The teachers' discussion provided answers to these questions. One teacher said, as her students were studying slopes, "What they don't know because of memorizing symbols with no picture in mind!" Teachers finally began to use both the graph and the symbolic representation as they explored some new dimension of algebra, "It took a few days to get used to reading the pictures, then to tack it back on to what we know." Midway through the series, teachers showed a tendency to abandon one perspective for the other, the old for the new or the familiar for the unfamiliar, rather than a combination of the two. In the later sessions, teachers said their work with the graphs had given them insight into their work with symbolic representations, "It makes all the sense in the world, it feels right—it's just another representation."

Teachers seem to have gone through a succession of stages. At first they were skeptical about anything that they were unfamiliar with—the symbolic was valid and central to the subject. As they began to learn about mathematics and could understand more broadly and deeply when using graphical representations, they began to think that all of algebra should be done graphically. It was only after additional reflection that they realized they were using the two forms of representation in complementary ways; they began to move toward a more balanced position, treating the two representations as complementary, each with strengths and weaknesses.

The Teachers and Algebra Project set out to reform the mathematical content and pedagogy of algebra. To achieve its goal, it built a reformed curriculum around the mathematical idea of function. Project staff are now examining whether teachers find the reformed content both learnable and acceptable. In the third and final year, project data will be analyzed and a final report on project activities completed. Preliminary analyses of project data suggest that while teachers do not learn mathematics from their coworkers, teachers occasionally learn new techniques and approaches to the teaching of mathematics from their coworkers. Those who were motivated to change their classroom practices often encountered skepticism and hostility from their less change-oriented coworkers. Where there were evident shifts in teachers' practices, teachers received strong administrative support and encouragement for their efforts. The project is supported by the Teaching/Learning of Algebra Working Group of the National Center for Research in Mathematical Sciences Education.

References

Harel, G. & Kaput, J. (1991). Conceptual entities in advanced mathematical thinking. The role of notations in their formation and use. In D. Tall (Ed.), *Advanced mathematical thinking*. Boston & Dordrecht: Kluwer.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards from school mathematics*. Reston, VA: Author.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects on different sides of the same coin. *Educational Studies in Mathematics,* 22(1), 1-36.

Review of NCRMSE Research

Generations of algebra students have found abstract algebraic expressions difficult to understand. There is a consensus within the mathematics community that functions—one form of algebraic expression—are among the most powerful and useful of mathematical notions. Most advanced mathematics courses are built around the use of symbolic representations of functions. First courses in algebra introduce content on the symbolic representation of functions and the manipulation of these symbolic representations.

Graphical representations of functions receive little attention in algebra classes. While research is sparse on how the use of graphical representations increases student understanding, those working with graphical representations believe that, for most students, such representations make functions easier to learn about and use. Creating graphical representations was once a time-consuming and cumbersome task for teachers. With today's computers and graphing calculators, teachers and students can create graphical representations of functions and transformations of these functions quickly and easily. These technological tools have the potential both to reshape the ways in which algebraic concepts are taught and to restructure the levels at which they are taught.

The National Center for Research In Mathematical Sciences Education began an effort to synthesize research on teaching, learning, curriculum, and assessment that focused on graphical representation of functions in 1988. It initiated its efforts with a conference that brought together researchers who were concerned about current issues in the domain, the impact of technology on the domain, and the integration of research on teaching, learning, curriculum, and assessment. Participants at the conference developed a common vocabulary with which to describe their work and a common agenda for future work. This led to the development of a series of papers on the content of the domain and what is known about student thinking, teacher thinking, teacher knowledge, classroom instruction, and curriculum that relates to the domain. Conference participants have continued their efforts as members of the NCRMSE Working Group on the Teaching/Learning of Algebra and Quantitative Analysis. The Working Group, chaired by James Kaput, University of Massachusetts-Dartmouth is described earlier in this newsletter. This review summarizes the papers of original conference participants, which consider the implications of graphing technology for mathematics educators and their students.

Three authors, Frank Demana of Ohio State University, Harold Schoen of the University of Iowa, and Bert Waits of Ohio State University, believe that graphing is a first step for learners as they build an understanding of the representation of functions. They examined the content that involved graphs in a typical mathematics textbook for each of the Grades 1-8. According to their analysis, about 3 percent of the textbook pages for Grades 7 and 8 contain graphing content; this amount is twice the amount found in any of the textbooks for the earlier grades. They concluded that students in Grades 1-6 have almost no experience constructing a graph of any kind. Nor do students encounter activities that ask them to make a global interpretation of a graph. The authors note that the NCTM Curriculum and Evaluation Standards for School Mathematics (1989) recommend the introduction of appropriate graphing activities very early (Grades K-4) and make these activities a major part of the curriculum at middle and secondary levels.

Author Sharon Dugdale, University of California-Davis, reviews her own and others' research on students' conceptions of functional relationships. Her review cites research evidence that students who view transformations of functions graphically and in the form of algebraic symbols perform better on non-standard questions than those who only view them in the form of algebraic symbols. She concludes that instructional methods that use multiple approaches need to be designed to improve students' development of graphical skills. While function-plotting software permits visual representations of algebraic functions to play a more substantial role in classes, according to Dugdale, any revised instructional approaches need to consider how students' perceptions change and how their ideas about graphical representations evolve as they acquire broader experience.

Skills, connections, and coordinations that are difficult for students to develop, according to Judit Moschkovich and Alan Schoenfeld of the University of California-Berkeley and Abraham Arcavi of the Weizmann Institute of Science-Rehovot, Israel, may seem trivial to those who have mastered them. Their chapter reports on a research and development program that is mapping the way students come to understand the complex domain of different symbolic representations of linear relations so that curricula can be constructed that help students deal with that complexity. Their work is based on an approach that defines understanding as making connections and analyzing "content domains" to determine the kinds of connections that competent practitioners make. In their view, pedagogy needs to move from its emphasis on procedural skills to an emphasis on making connections and developing and understanding if it is to plan algebra content for students that is mathematically interesting.

A historical account of the evolution of teacher and textbook thinking about functions is provided by Thomas Cooney and Melvin Wilson of the University of Georgia. They conclude that there has been a movement in secondary mathematics during the 1990s toward an increasing emphasis on the concept of function. In their words, "The emphasis on functions as a unifying mathematics concept, as a representation of real-world phenomena, and as an important mathematical structure remains central to contemporary discussions." They note that the definition of a function has changed over the last decade and that technology may become a major factor in determining how school mathematics will treat functions in the future. Their review of relevant research leads them to conclude that little is known about the interaction between teachers' knowledge and beliefs about functions and the extension of these interactions to their classroom practice. The authors caution future researchers who would examine teachers' knowledge and beliefs to appreciate the importance of "the context in which knowing and believing occurs."

While knowledge about what teachers know and how they think about algebra is fragmentary, according to F. Alexander Norman of the University of Texas at San Antonio, some inferences can be drawn from research on students' understanding of algebra. He suggests that these inferences can be used by researchers to begin a comprehensive investigation of teachers' knowledge of graphing and functions. His review of research includes categories on students' understanding of functions, graphs, and multiple representations. It concludes with a series of research questions that need to be addressed and a description of a long-term study that deals with some of the questions. A few of the questions are: Are the cognitive learning processes of teachers and the specific knowledge required for them different from those of students? How important do teachers perceive that function concept to be? In view of the different emphases suggested by the NCTM Curriculum and Evaluation Standards for School Mathematics (1989), how might teachers' beliefs about functions and graphs affect their acceptance of a mathematics curriculum with a different orientation? What are teachers' views on the importance of introducing the notion of functions via multiple representations?

In her chapter on classroom instruction, Carolyn Kieran of the University of Quebec at Montreal traces work on students' perceptions of functions. Relying on research completed in an international sample of countries, she emphasizes the differences in perceptions among age and ability groups of students and the implications these hold for teachers. Sketching and interpreting graphs, while routinely taught in The Netherlands and England, receive little attention in North American schools. Further, when U.S. students tabulate, plot, or read values from graphs, they seldom have the opportunity to apply their skills to practical situations. She points out, in a historical narrative, that graphs were used to represent early conceptions of functions. Author Kieran concludes her chapter with descriptions of recent projects using technology-supported environments that involve graphs and functions. The descriptions fit into three categories: (a) activities that do not require a knowledge of algebra; (b) activities that are included in first-year algebra courses; and (c) activities with students who have completed at least one course in algebra.

She concludes that a "tide of change" is evident and that the "newer" approaches to graphing have three major focal points, interpreting global features of graphs, using basic families of graphs to explore the roles of the parameters of the algebraic representation, and using graphs as a tool for problem solving in applied settings. The teachers involved in the projects she described reported positive effects of the new approaches on student motivation.

In their chapter on the curricular implications of the graphical representation of functions, Randolph Philipp of San Diego State University and William Martin and Glen Richgels of the University of Wisconsin-Madison make the point that "for most of this century, curricular decisions regarding the use of graphs and functions have been made without the benefit of research." In their historical perspective, they trace the introduction of algebra during the first half of the 19^{th} century at Harvard, Yale, and Princeton. Algebra was moved from a college to a high school course with a few changes in its content later during the same century, according to their account. They cite a 1926 survey in which "all high school teachers, except mathematics teachers, believed that more students ought to be enrolled in their courses" and a recommendation of the same ear that would have admitted only 25 percent of high school students to algebra classes. They state that a main implication of the "graphical representation capabilities of computers and calculators is that solutions to interesting and realistic problems are accessible, even without well-developed manipulative skills." With examples, they show that students who have dropped out or been excluded from mathematics may have greater access to mathematics with a graphically-oriented curriculum. Such a curriculum, they note, would require changed approaches to teaching, classroom materials, and assessment.

While students have found algebra difficult to understand, "our only educational solution to date has been demographic—to eliminate…students from the responsibility for learning it," says James Kaput of the University of Massachusetts-Dartmouth. He calls for proleptic research, research that deliberately anticipates the future, to remedy what "we [are now seeing as] a severe conservative bias in mathematics education research relative to technological change…that is likely to endure." He covers alternatives to present-day approaches that would place algebra in the middle grades, that would teach algebra concepts to students who have not yet mastered arithmetic in its several aspects, that would pay significantly more attention to linking experiences to formal mathematics, that would incorporate simulations that provide "continuous, real-time feedback in several experiential dimensions." Finally, he points to additional research that should be undertaken to anticipate the mathematics of the future.

In his summary chapter, Steven Williams of Washington State University concludes that good teaching, "the careful choice of mathematical tasks coupled with efforts to establish an environment for exploration, conjecture, and dialogues, [has] been shown in a number of studies to be successful in leading students to understanding functions and their graphs." Assuming the view of learning as enculturation, he calls for an in-depth analysis of the content domain or the specification of the conceptual field for functions and graphs. Technological tools, he cautions, must be seen as tools and not "as creators of knowledge." What technology can do, he says, must be linked to human activity and human concerns; careful anthropological studies of how functions and their graphs are used by various cultures in their everyday practices could help to define the content domain. While the complexity of the domain makes doing research on the cognitive processes used by students difficult, he points readers to research strategies that effectively are uncovering the understanding of students. As a basis for further modeling of the understanding, learning, and teaching of graphs and functions, however, he points out that we need more of these careful analyses. Researchers, he says, have yet to assess which functions and graphs are “put to use in the mathematical cultures…our students will need to join.”

The papers from which the above comments were drawn form the basis for *Integrating Research on the Graphical Representation of Functions*, a book edited by Thomas A. Romberg, director of NCRMSE, and Elizabeth Fennema and Thomas P. Carpenter, associate directors of NCRMSE. The editors are professors of mathematics education at the University of Wisconsin-Madison. The book will be published by Lawrence Erlbaum Associates, Icorporated, during the spring of 1993.

Reference

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.