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Developing Mathematical Reasoning within the Context of Measurement

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OUR purpose in this paper is to describe how one group of students developed personally meaningful ways to reason mathematically within the context of measurement. To clarify our viewpoint, we present episodes taken from a first-grade classroom in which we conducted a four-month teaching experiment. (The authors of this paper were all involved in the teaching experiment. The fourth author was also the classroom teacher.) One of the goals of the teaching experiment was to develop instructional sequences designed to support first graders' construction of meaningful understandings for (1) measurement and (2) mental computation and estimation strategies for numbers up to 100. A primary focus when developing the instructional sequences was to support students' multiple interpretations of problem situations. These interpretations would then serve as the basis for classroom discussions in which students explained their mathematical reasoning. Our intent in presenting the episodes is not to offer examples of exemplary teaching. It is, instead, to provide a setting in which we can examine measurement as a context for supporting students' construction of sophisticated ways to think and reason mathematically. (See also article 10, by Artzt and Yaloz, in this volume.)

In the following sections of this paper, we first outline the intent of the instructional sequences developed in the course of the teaching experiment. Against that background, we then describe episodes from the classroom that highlight students' ability to reason mathematically while investigating issues related to measurement.

THE INSTRUCTIONAL SEQUENCES

The first sequence of the teaching experiment dealt with measurement. The second one built on the measuring activities to support students' construction of mental computation and estimation strategies for reasoning with numbers to 100. In the instructional sequence that dealt with measurement, our initial goal was that students might come to reason mathematically about measurement and not merely measure accurately. This approach differs significantly from many that are frequently used in American schools in that the focus was on the development of understanding rather than the correct use of tools. In particular, we hoped that the students would come to interpret the activity of measuring as the accumulation of distance (cf. Thompson and Thompson 1996). For instance, as the students were measuring by pacing heel-to-toe, we hoped that the number words they said as they paced would each come to signify the measure of the *distance* paced rather than the *single pace* that they made as they said a particular number word (e.g., saying "twelve" as students paced the twelfth step would indicate a distance that was twelve paces long instead of just the twelfth step). Further, our intent was that the results of measuring would be structured quantities of known measure. If this were so, students would be able to think of a distance of, say, 20 steps that they had paced as a quantity itself composed of two distances of ten paces, or of distances of five paces and fifteen paces. In doing so, it would be self-evident that whereas distances are invariant quantities, their measures vary according to the size of the measurement unit used.

As we shall see, measuring with composite units became an established mathematics practice in the course of the teaching experiment. Initially, the students drew around their shoes and taped five shoe prints together to create a unit they named a *footstrip*. Later, in the setting of an ongoing narrative about a community of Smurfs, the students used a bar of ten Unifix cubes to measure. These instructional activities evolved into measuring with a strip that was the length of 100 Unifix cubes. This in turn made it possible for the students' activity of measuring to serve as the starting point for the second instructional sequence that addressed mental computation and estimation with two-digit numbers. Our immediate concern was not merely that students would acquire particular calculational methods. Instead, we also focused on students' construction of numerical relationships that are implicit in these methods. This view shifts the importance from calculational strategies per se to the mathematical interpretations and understandings that make the use of flexible strategies possible.

CLASSROOM EPISODES

In the classroom in which we worked, the teacher often attempted to initiate shifts in the level of classroom discourse so that what was done mathematically subsequently might become an explicit topic of conversation. As part of this process, the teacher encouraged students to explain their reasoning by grounding their explanations in the use of tools or by drawing pictures on the white board (cf. McClain and Cobb in press). For example, in discussing solutions to a simple task such as *There are 11 cats and 3 dogs. How many more cats than dogs are there?,* the students drew tallies on the white board to represent the cats and the dogs and then explained their reasoning with reference to the tallies. This helped the students communicate their thinking and often resulted in their solution methods becoming topics of conversation and investigation.

The instructional activities used in the teaching experiment were typically posed in the context of an ongoing narrative. To accomplish this, the teacher engaged the students in a story in which the characters encountered various problems that the students were asked to solve. The narratives both served to ground the students' activity in imagery and provided a point of reference as they explained their reasoning. In addition, the problems were sequenced within the narratives so that the students developed increasingly effective measurement tools with the teacher's support. Further, the narrative supported the emergence of tools out of students' problem-solving activity.

The first narrative involved a kingdom in which the king's foot was used as the unit of measure. The initial instructional activities involved measuring by pacing. The first tool the students developed to resolve a problem was called a *footstrip* and consisted of five shoe prints taped together heel-to-toe. A number of mathematically significant issues emerged during the first part of the sequence, including that of describing distances measured with footstrips (e.g., 5 footstrips versus 25 paces).

The second narrative developed during the measurement sequence involved a community of Smurfs who often encountered problems that involved finding the length or height of certain objects. The teacher explained that the Smurfs' food came in cans and that they decided to measure objects by stacking cans to the height of the object to be measured. In the classroom, the students used Unifix cubes as substitutes for cans and measured numerous objects for the Smurfs such as the height of the wall around the Smurf village, the length of the animal pens, and the depth of the water in the river. After several measuring activities, the teacher explained to the students that the Smurfs were getting tired of carrying around the large number of cans needed for measuring. The students agreed that this was cumbersome and discussed alternative approaches. Several suggested iterating a bar of cubes (cans), presumably influenced by the prior activity of measuring with the footstrip. This discussion seemed to influence their decision to measure with a bar of ten cubes that they called a *Smurf bar.*

When measuring with the Smurf bar, all the students measured objects by iterating the bar along the length of the item to be measured and counting by tens. However, some students counted the last cubes of the measure within the last iterated decade (see fig. 8. 1).

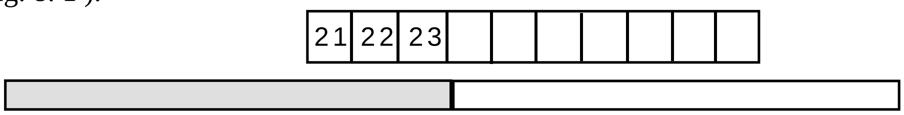

Fig. 8. 1. Counting 21, 22, and 23 within the second decade

Solutions of this type became the focus of discussions as can be seen in an incident that occurred two weeks after the measurement sequence began. The teacher had posed the following task: *The Smurfs are building a shed. They need to cut some planks out of a long piece of board. Each plank must be 23 cans long. Show on the board where they would cut to get a plank 23 cans long.* Students had been given long pieces of adding machine tape as the board and were asked to use a single Smurf bar to measure a plank the length of 23 cans. Angie was the first student to share her solution process with the class. She showed how she had measured a length of 23 cans by iterating the bar twice and then counting 21, 22, and 23 beyond the second iteration. When she finished, Evan disagreed.

Evan: I think it's 33 because 10 *(place bar down as in figure 8.2a), 20 (moves bar as in figure 8.2b),* 21, 22, 23 *(points to the cubes within the second iteration, thus measuring a length that was actually 13 cubes).*

Angie. Um, well, see, look, if we had 10 *(moves bar as shown in figure 8.2a)* that would be 11, 12, 13.

Evan: But it's 23.

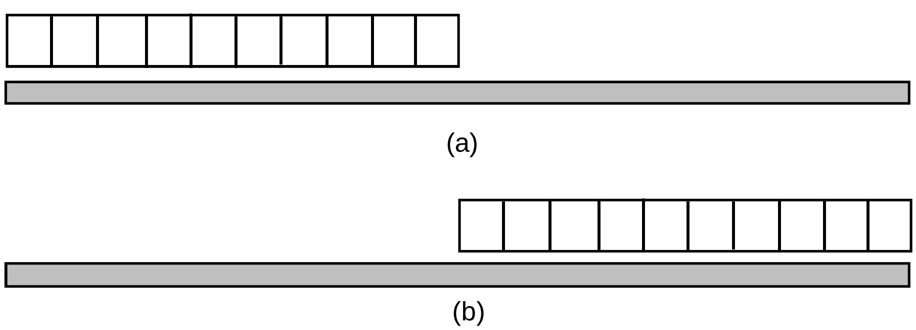

Fig. 8.2. (*a*) Showing the first iteration in measuring 23 cans; (*b*) showing the second iteration in measuring 23 cans

At this point, Evan and Angie appeared to be miscommunicating. Although Angie appeared to understand how Evan found a length that differed from hers, she was unable to explain her reasoning to him. This miscommunication continued as Andy explained why he agreed with Evan. The teacher then asked both Evan and Angie to share their solution methods again.

Teacher: Let's be sure all the Smurfs can understand 'cause we have what Angie had measured and what Evan had measured. We need to be sure everybody understands what each of them did, so Evan, why don't you go ahead and show what it is to measure 23 cans.

Evan: Ten (places bar as shown in figure 8.2a), 20 (moves bar as shown figure 8.2b). *(Pause)* I changed by mind. She's right.

Teacher: What do you mean?

Evan: This would be 20 *(points to end of second iteration).*

Teacher: What would be 20?

Evan: This is 20 right here *(places one hand at the beginning of the "plank" and the other at the end of the second iteration).*

Teacher: So that where your fingers are shows a plank that would be 20 cans long? Is that what you mean? Any questions for Evan so far?

Evan: Then if I move it up just 3 more. There. *(Breaks the bar to show 3 cans and places the 3 cans beyond 20.)* That's 23.

It appeared that in the course of reexplaining his solution, Evan reflected on Angie's method and reconceptualized what he was doing when he iterated the bar. Initially, for Evan, placing the Smurf bar down the second time as he said "20" meant the twenties decade. Therefore, for him, 21, 22, and 23 lay within the second iteration. However, he subsequently reconceptualized "20" as referring to the distance measured by iterating the bar twice and realized that 21, 22, and 23 must lie beyond the distance whose measure was 20. This type of reasoning was supported both by the teacher's asking Evan to explain his method so that everyone would understand and by Evan's counting the cans that he iterated when moving the bar (i.e. , the measure of the first two iterations was 20 because he would count 20 cans). As a consequence, for Evan, the activity of measuring with a Smurf bar now appeared to be about structuring space. He broke off three cubes from the Smurf bar to show 23 instead of working within the third iteration to find 21, 22, and 23.

As the episode continued, Angie continued to explain her thinking:

Angie: I have a way to help Andy because he thinks like Evan before he changed his mind. You see like this is 10, but you know that, right? *(Andy nods in agreement as Angie places down the first iteration.)*

> 11, 12, 13,..., 19, 20 *(moves bar to second iteration and counts each cube individually, pointing to the cubes as she counts. She then moves the bar to the third iteration as she continues counting.)* 21, 22, 23. So it goes two tens and three more.

Here, in grounding her explanation in the counting of the individual cans that composed the Smurf bar, Angie attempted to clarify her reasoning to Andy. As her explanation indicates, measuring involved the accumulation of distance in that iterating the bar while counting by tens was a curtailment of counting individual cans. As a consequence, it was self-evident to her that she needed to measure beyond the second iteration to specify a length of 23 cans.

It is important to note that the teacher's overriding concern in this episode was not to ensure that all the students measured correctly. In fact, the teacher frequently called on students who had reasoned differently about problems in order to make it possible for the class to reflect on and discuss the quantities being established by measuring. Her goal was that measuring with the Smurf bar would come to signify the measure of the distance iterated thus far rather than the single iteration that they made as they said a particular number word. Her focus was therefore on the development of mathematical reasoning that would make it possible for the students to measure correctly with understanding.

After the students had measured several planks and other items with the Smurf bar, the teacher explained that the Smurfs decided they needed a new measurement tool so they would not have to carry any cans around with them each time they wanted to measure. In the ensuing discussion, several students proposed creating a paper strip that would be the same length as a Smurf bar and marked with the increments for the cans. Students then made their own *ten-strips* and used them to solve a range of problems grounded in the Smurf narrative. During a discussion about the meaning of measuring by iterating a ten-strip, the teacher taped several of the students' strips end-to-end on the white board to show successive placements of the strip. In doing so, she created a *measurement strip* 100 cans long. Crucially, this new tool emerged from, and was consistent with, the students' current ways of measuring. As a consequence, they could all immediately use prepared measurement strips with little difficulty. They simply placed the strip along the dimensions of the object to be measured and counted along the strip by ten or sometimes by five.

All the instructional activities we have discussed thus far involved measuring the lengths or heights of physical objects. The transition from the measurement sequence to the mental computation and estimation sequence occurred when the students began to use the measurement strip to reason about the relationship between the lengths or heights of objects that were not physically present. One of the first instructional activities in the mental computation and estimation sequence involved an experiment the Smurfs were conducting with sunflower seeds. The teacher explained that the Smurfs typically grew sunflowers that were 51 cans tall. However, in one of the experiments the sunflowers grew only 45 cans tall. Students were then asked to find the difference in heights and were given only a measurement strip. As a consequence, they could not create objects 45 cans and 51 cans long to represent the sunflowers, but instead had to reason with the strip.

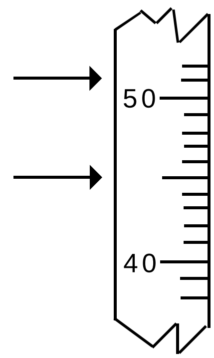

Fig. 8.3. Measurement strip marked to show 51 and 45 cans

Students first worked in pairs to solve the task and then discussed their solution in the whole-class setting. The teacher began by placing a vertical measurement strip on the wall and asking students to mark both 51 cans and 45 cans (see fig. 8.3).

An issue that emerged almost immediately in the discussion was that of whether to count the lines or the spaces on the strip.

Teacher: Think about how you would show us how much shorter that seed *(points to the 45)* grew than the regular seed. Preston, how would you show that?

Preston: Here *(points to 51)* all the way down to here *(points to 45)* would be seven.

Teacher: Can you show me the seven?

Preston: Here is 51 and here is 45 and here is 1, 2, 3, 4, 5, 6, 7 *(points to lines as he counts).*

Pat: I have a question. You are supposed to count the spaces, not the lines.

At this point, the teacher asked Pat to explain why he thought you use the spaces and not the lines.

Pat: The cans of food are bigger than the lines, and you are trying to figure out how many cans, not lines.

Teacher: So when you say *space*, you think of this space as a can of food *(points)?*

Pat: And that's how much, and you're trying to figure out how much that is.

For Pat, reasoning with the measurement strip was related to the prior activity of measuring with a Smurf bar. As a consequence, the spaces signified Unifix cubes or cans for him. In contrast, reasoning with the strip did not appear to be grounded in prior activity for Preston, and he was simply trying to figure out a way to use it to solve the task at hand. However, Pat's explanation led Preston to modify how he reasoned with the strip. This is evidenced by the fact that Preston asked if he could use a can or cube to help him solve the task. He placed a single cube on the measurement strip and iterated the spaces between 45 and 51 to arrive at the answer of six rather than seven.

> *Preston:* I changed my mind. I changed my mind because the lines are smaller than the squares [cubes].

Immediately after the exchange between Pat and Preston, Andy gave an explanation that involved reasoning about the quantities in a different way.

> *Andy:* If you went from 50 down five, you'd get to 45 cans. Think 5 less than 50. But you are really one more, so it's six, since it's I more than 50.

Andy's explanation indicates that for him, as for Pat, 45 and 51 signified distances from the bottom of the strip measured in cans. The task for him was to find the difference between these two quantities, and he did so by reasoning with the strip. We would, in fact, argue that the strip supported the shift he made from a counting to a thinking strategy solution in which he reasoned that 50 to 45 was five, so 51 to 45 is six.

It is important to note that the solution method offered by Andy fit with the teacher's pedagogical agenda of supporting students' development of increasingly sophisticated strategies. However, the teacher was also aware of differences in her students' reasoning and did not want to create a situation where students simply imitated strategies that they did not understand. As a result, she continued to acknowledge the differing ways that students reasoned about tasks while highlighting solution methods that fit with her agenda. This served to support proactively the development of the students' mathematical reasoning. The diversity in students' reasoning as they used the measurement strip can be seen in an episode that occurred one week later. The students were first asked to work in pairs to find their height and their partner's height. Once the results had been recorded on the white board, the teacher asked the students to find the difference in heights for each of the pairs. Students were not instructed how to solve this task, and several ways of reasoning emerged.

When students returned to whole-class discussion, the teacher asked them to share ways that they found the difference between Luis's and Andy's heights, which were 67 cans and 72 cans respectively. She then marked both 67 and 72 on the measurement strip. Mari first explained that she and her partner had found a difference of six cans. Mae appeared to anticipate how they had reasoned and asked:

> *Mae:* Did you count the 67 or did you go on to 68?
>
> *Mari:* We counted the 67.

At this point, Mari went to the measurement strip that was posted on the wall and took a Smurf bar to try to determine exactly how many cans would fit between 67 and 72. For her, it seemed essential that she actually measure with cans or cubes. As she did so, Andy offered a more sophisticated solution.

Andy: I knew that 7 plus 3 would get us, uh, 67, plus 3 would get us to 70. I was two more than 70, and I knew that 3 and 2 was 5.

In contrast to Andy and Luis's going-through-ten solution, Hanna explained:

Hanna: I just counted on my fingers, and I agree with Luis and Andy.

As other students discussed the problem, Mari continued to build a Unifix cube bar the length of the difference. When she finished, she counted the five cubes in the bar.

Teacher: Mari, is that the same five as Andy?

Andy: It's the same five. Mari did it by ones and I did it by 3 and 2)cause it's easier to go 3 more.

In this particular episode, a variety of ways of reasoning emerged while discussing solutions to the task. As before, the teacher continued to acknowledge the differing ways of reasoning. This diversity continued to support students' ability to construct personally meaningful ways to reason mathematically.

Later in the instructional sequence, an empty number line—a line without any markings—was introduced as a means of reasoning about quantities while also providing a way for students to record their thinking. The empty number line evolved from activities with the measurement strip and was initially used to show the relationships between quantities. For instance, the teacher asked students to show *about* where 50 cans would be. They then compared that to *about* where 30 cans might be. In the process of whole–class discussions, students began to use the empty number line as a means of explaining how they reasoned about the relative magnitude of certain quantities. The emphasis in these activities was not on marking numbers in an exact place, but on using the relative positions to let the number line represent how they had reasoned about the task. In other words, students might reason that 50 is about in the middle since it is half of 100. Thirty would be a little more than halfway between the beginning of the empty number line and 50. As the sequence progressed, changes in the ways the students reasoned with the empty number line became evident. For instance, on the third day that the students used the empty number line, the following task was posed: *Maria has 87 cookies in a box. How many will be left if she eats 18 of them?* After the students had worked individually, Angie shared her solution method at the white board during whole-class discussion by first marking eight "jumps" of ten and a "jump" of seven to show 87 (see fig. 8.4). She then worked backwards from the 87 to take away eighteen cookies by removing the seven, a group of ten, and one more (see fig. 8.5). She indicated which

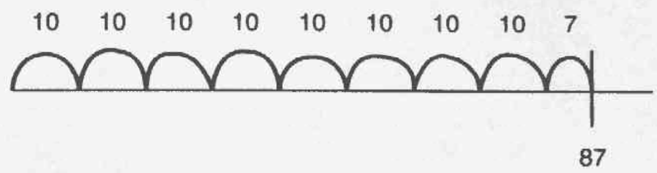

Fig. 8.4. Angie's explanation of 87 cookies

cookies she took away by circling the eighteen that had been eaten, as shown in figure 8.5.

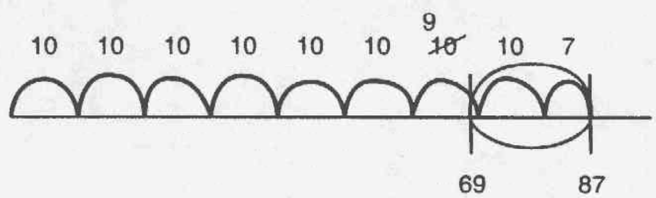

Fig. 8.5. Angie's explanation of 87 cookies and 18 are eaten

When she finished, Pat asked, "Why did you take the 1 from the 9?"

Angie: I want to take away that 7 *(points)* and that 10 *(points)* and 1 more 'cause if I just took away the 10 and then 7 that would be 17, right? And it says here you want to take 18, so I need one more.

Pat: Oh, I understand.

In response to the teacher's request for other solutions, Michael simply marked 87 on the empty number line and then drew a jump of ten to 77 and a jump of eight to 69 (see fig. 8.6).

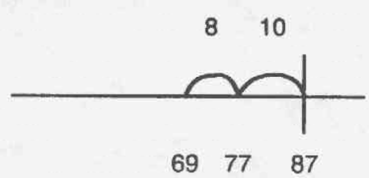

Fig. 8.6. Michael's solution for 87 cookies take away 18 cookies

In both of the examples above, the students' thinking seemed to reflect their prior activity of measuring with the Smurf bar and reasoning with the measurement strip. In Angie's case, it was important to first create the 87 by iterating the tens, whereas Michael was able to take the 87 as a given quantity.

To present the next task, the teacher showed a picture of 17 cups and 48 saucers and asked the students to determine how many more cups were needed so that there would be the same number of cups and saucers. Mari explained her solution at the white board by first drawing jumps of ten and seven and marking 17. She then continued to draw jumps of ten until she reached 47 and finally drew a jump of one to get to 48 (see fig. 8.7).

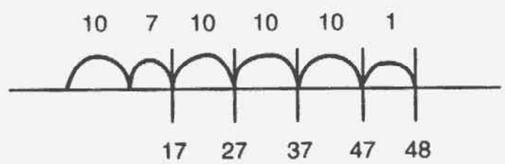

Fig. 8.7. Mari's solution to the cups and saucers problem

In discussing Mari's solution method, students questioned how what she had drawn helped solve the task.

Pat: How can this be 48? When you put the ten and seven together you only have 31.

Mari: I mean this whole thing to be the saucers, and I'm using the 17 twice. I'm using the 17 to be cups and then put it with all this *(points to entire drawing)* to make 48 saucers.

Pat: Yeah, but I can't tell them apart. How can you tell these are cups and the whole thing is saucers? I don't think you can use the same thing twice.

Hanna then volunteered that she could help Pat understand Mari's reasoning.

Hanna: I know that she has 17 cups there *(points to* 17) and altogether that's the saucers, but you have to count to 17 to get the saucers.

At this point, the teacher asked Mari to show where the cups were on her number line. In response, Mari drew the arc from the beginning of the number line to 17 and labeled it "17" as shown in figure 8.8. She then continued by drawing an arc above the line, explaining that this showed the 48 saucers (see fig. 8.8). The teacher then asked

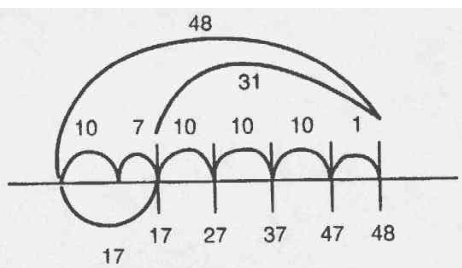

Fig. 8.8. Notating on Mari's number line

the class, "Where are the extras, the cups we need?" Evan came to the white board and marked an arc on Mari's line that he labeled "31" (see

Thus for Evan, as for the other students, Mari's number line came to signify a relationship between quantities of cups and saucers. Further, the drawing served as a means of argumentation throughout the exchange. As a consequence, the discussion focused on how Mari had interpreted the problem rather than merely the calculational steps she took to produce an answer. The next student to explain his solution, Trent, took 17 as a given quantity and drew jumps from 17 to 48 to find how many more cups were needed (see fig. 8.9).

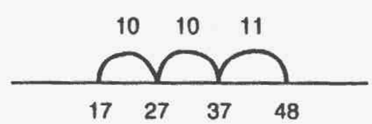

Fig. 8.9. Trent's solution

The teacher asked if someone could look at Trent's drawing and explain what he was thinking.

John: You need to go from 17 to figure how many more to make it equal.

Although John did not refer to cups and saucers, his explanation indicates that Trent had calculated with the number line in order to equalize his two quantities.

These episodes indicate the value of classroom discussions in which students not only explain their own reasoning but also reason about others' solution methods. As a result, the solution process, not the answer, is what is valued. Students then come to understand the obligation to be able to represent their thinking so that others might understand. In this way, the drawings provide an opportunity for students to reflect on and discuss various ways of reasoning about the situation. It is in the course of such exchanges that students come to understand the importance of representing their thinking so that it might be comprehensible to others. These representations then make it possible for students to reflect on and compare not just different calculational processes but different ways of interpreting and reasoning about situations.

In examining the students' written work for this particular task, it is interesting to note that they used a variety of methods in addition to those shared in the whole-class discussions. For example, Susan found the difference between 17 and 48 by first making a jump of 1 to 18 and then making jumps of ten until she reached 48 (see fig. 8.10a). Mae, however, found the difference by making a jump of 20 to 37 and then a jump of eleven to 48 (see fig. 8.10b).

For her part, Hanna first made a sequence of jumps to reach 48 and used these to find the difference between 48 and 17 (see fig. 8.11a). Luis's solution did not involve the use of the number line. Instead, he wrote a series of number sentences to express his reasoning, as shown in figure 8.11b.

These examples indicate that by the end of the teaching experiment, the students had developed a range of personally meaningful ways to reason about quantities. The teacher's role in supporting discussions that focused on the meanings that the students' records of their thinking had for them was crucial to this process.

CONCLUSION

In this paper, we have highlighted students' mathematical reasoning while focusing specifically on measurement. In doing so, we have noted the importance both of discussions in which students explain and justify their thinking and of carefully sequenced instructional activities. These interrelated aspects of instruction play an essential role in supporting students' development of powerful ways of reasoning. As we have noted, the intent of the instructional sequence on measuring was that students might come to

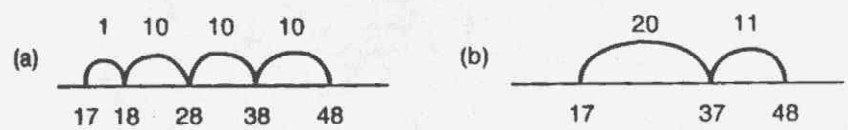

Fig. 8.10. (*a*) Susan's solution; (*b*) Mae's solution

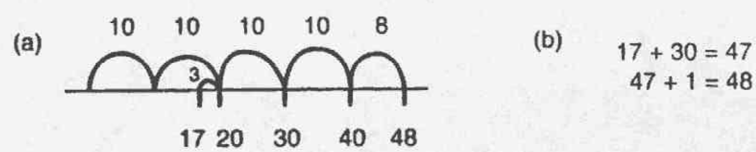

Fig. 8.11. (*a*) Hanna's solution; (*b*) Luis's solution

reason matheniatically about space and distance, and not merely measure accurately. In other words, students would be able to reason quantitatively about their measuring activity instead of simply using measurement tools correctly. This reasoning then served as the starting point for the second instructional sequence, which aimed to support their construction of relationships between numbers to 100. In addition, students seemed to reconceptualize their understanding of what it means to know and do mathematics in school as they solved tasks and discussed their reasoning. The crucial norm that became established was that of explaining and justifying solutions in quantitative terms. We find this significant because the students' reconceptualization of mathematics went hand in hand with their development of increasingly sophisticated way of reasoning. In particular, preliminary analysis of the data indicates that students who, at the beginning of the classroom teaching experiment, were unable to reason quantitatively with numbers up to 20 were, by the end of the experiment, able to reason in a variety of ways with numbers to 100.

In our view, the students' ability to act as increasingly autonomous members of the classroom community and their development of powerful ways of reasoning were reflexively related. Initially in this classroom, the teacher judged the value of students' contributions. By the end of the teaching experiment, the students could make these judgments and justify why their contributions were important. This is significant pedagogically and mathematically because it constitutes a change in the way many teachers perceive both their role and that of their students. This was an important part of the culture established in the project classroom and contributed to the students' development of mathematical power.

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