The Shape of Fairness

Elizabeth Penner and Richard Lehrer

Shape and form are often used as mathematical models of situations. For example, light travels in a line or the shadow cast by a person is related by similar triangles to that cast by a flagpole. Yet despite the common use of mathematical models in the sciences and in design professions, children rarely have the opportunity to participate in this form of mathematical thinking. In this article, we describe how first and second graders modeled a "fair" playing space in a game of tag called "Mother, may I?" The children modeled the playing space by using a succession of different forms, such as lines and squares, to represent a fair game, discovering along the way the properties of each of the forms that made them less-than-ideal models of fairness. Participation in the game gave the children many opportunities to think about important concepts in measuring length and the idea of using form to model a situation.

To play "Mother, may I?" one child was chosen as "mother" to give commands to the rest of the players, who are called "movers." The commands tell the remaining players how to move in the game. Mother could ask the class to take baby steps, giant steps, turn steps, and foot-to-foot steps or to leap frog. The children agreed on the action involved in each type of step. This agreement led them to think about different types of measurements, such as those for lengths or turns, and different units of measure, such as baby and giant steps. The goal of the game is to move toward and tag Mother. As we describe, the children discovered that if the players did not all start at the same distance from Mother, then the game was not fair. Much of the class time was used to find a form for the playing space that was fair to allow all players to start at the same distance from Mother.

Elizabeth Penner is a teacher of six-, seven, and eight-year-old children at Sugar Creek Elementary School in Verona, WI 53593. She is interested in helping children build mathematical understanding through modeling. Richard Lehrer, rlehrer@facstaff.wisc.edu, teaches courses in cognition and instruction at the University of Wisconsin–Madison, Madison, WI 53706. He investigates how children develop an understanding of mathematics and science in classrooms designed to support inquiry and sustained effort.

First Models

In the first game, Sam played Mother. The children arranged themselves spontaneously along a wall in relation to Sam so that, as Carly said, "We should all be the same distance from Sam." **Figure 1** shows the resulting configuration.

When the children began to play, some immediately protested that the game "wasn't fair," citing the advantage enjoyed by the players at the end closest to Sam. Their teacher, Ms. Penner, asked the children how they might make the game fair. Several children suggested that fair would be "in the middle." Sam then moved parallel to his classmates and stopped at a point approximately at the midpoint of the line. This second configuration is shown in **figure 2**.

Resuming play, the movers found that their lot was generally improved, but protest came from children at either end of the line, especially those who had enjoyed an advantage in the previous round. To prompt the students to reflect on the situation, their teacher asked the children to draw "the shapes our bodies are making." The most common representations were drawings that preserved the images of people, using stick figures or dots in the relative configuration shown in **figure 2**.

FIGURE 1

First configuration of Mother and movers

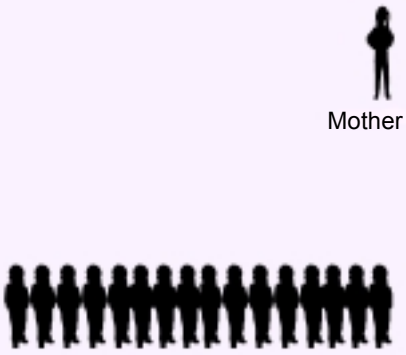
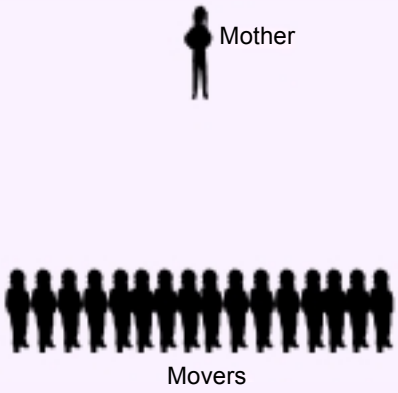


FIGURE 2

Second configuration of Mother and movers



Model Revisions

In the second lesson, Penner displayed children's representations of their first and second conjectures about a fair playing space and asked whether they noticed any patterns. John said, "We are all moving up towards Mother and in." For other children, moving up and in meant that the shape was "looking like a triangle." Seizing on this description of a form, the teacher began to help children make a transition from drawings to diagrams. She drew a line segment and a point to represent the first two models, asking whether her drawing showed the configurations of Mother and the movers, even though it did not use representations of particular children. The children agreed that both their drawings and their teacher's diagram showed important aspects of what they had done with their bodies. This discussion was the first step toward using spatial forms, that is lines and points, to represent the situation.

Pattern Blocks as Models

Recalling the talk about triangles, one student pulled out an equilateral triangle block from the class bucket of pattern blocks. Using this pattern block as a model, Sandy drew an outline of the triangle, putting Mother at its apex and dots representing movers along the base. The triangle captured the essential elements of the second configuration, just as the teacher's representation with a point and line segment had done. The triangle also highlighted the comparatively longer distances at the endpoints.

Penner asked whether other shapes in the bucket of pattern blocks might be fairer to use in the game. Some children proposed a trapezoid, suggesting that it had both the line of the second configuration and the "sides" of the triangle. The teacher asked the children to trace the triangle and trapezoid shapes on graph paper and to explore game configurations that might be fair. The grids on the graph paper provided the children with a measure for establishing relative distances between Mother and the movers in each form. The children quickly discovered, however, that no matter where they placed Mother, some movers would always be at an advantage. An important result of this day's investigation was the children's emerging understanding that the purpose of a model is not to replicate the world but rather to represent the essential attributes of the players, in this instance, their relative positions.

Reconsidering Shape

The next lesson began with a discussion of the nature of shape because Penner wished to enlarge the repertoire of forms that children could consider as potential models. On the chalkboard, she drew different shapes ranging from lines to polygons to curves, both open and closed. Because the children were convinced that shapes needed to "connect," they ruled out lines and open curves, although mathematicians might not!. A circle drawn on the chalkboard posed a problem. The children had learned early in their schooling that a circle was a shape, yet they could not see at which point it "connected." This issue led to much discussion about lines and connections, during which Elissa suggested that a line was "like a path leading you somewhere with a beginning and an end." John expanded on this notion: "Oh I get it," he as he made circular motions with his arm, "a circle is a path that keeps going around and around." The children concluded that even though they could not see the exact point at which a circle "connected," it was nevertheless a shape, according to the rationale suggested by John.

Groups of children explored connection and closure further by forming outlines of shapes with their bodies, and several children noticed that they could step in and out of the space defined by their bodies. They modified their ideas about the need for shapes to connect to include the idea of shape as defining an inside and outside.

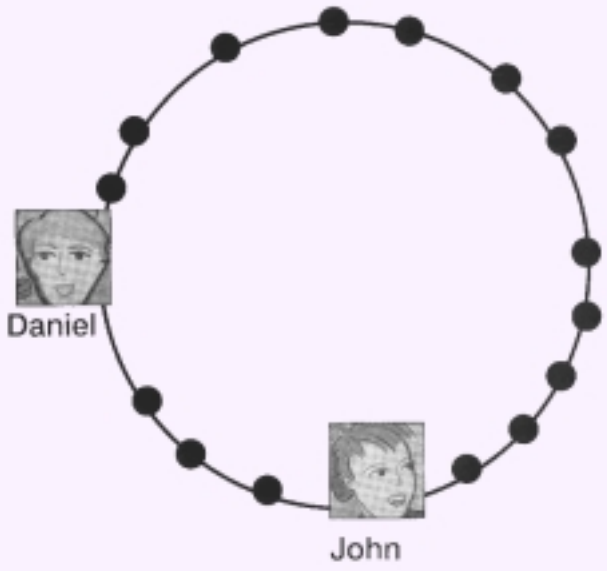
Considering the nature of shape expanded the pool of shapes that could be used in the game to include forms other than those found in the bucket of pattern blocks. The conversation about inside and outside led several children to propose that a fair space could be considered as a point inside a form; the point would represent Mother, and the movers would be arranged on the perimeter of the form. Some children decided that because a square's sides were all the same length, if Mother was placed in the center of the square, no mover would have an advantage. When students investigated this conjecture by tracing squares on graph paper, they found, to their surprise, that the distances from the center to the corners were not the same as those from the center to a point on the side directly opposite the center. They also found that the diagonal of a square was not the same length as any of its sides.

Do Circles Model Fairness?

Jennifer proposed a circle to resolve the problem of the square: "If Mom were in the middle, then the movers could go around her." When Penner asked what made the circle fairer, Jennifer replied, "We can all play, and we are all the same distance from Mother." Many children agreed with Jennifer, Penner decided to move from the classroom to the playground. Using a

FIGURE 3

Pacing to find the center of the circle



painted circle, she asked children how they knew that everyone was the same distance from mother. The children simply replied that "it looked that way" to them. Penner then asked Jennifer to put herself in the center of the circle. After much jostling and helpful advice, the children, especially those who perceived that they were further from Jennifer than their classmates, decided that they would need to find a more reliable way to locate the center of the circle.

Daniel and John proposed a solution. Positioning themselves approximately 90 degrees from each other, they began pacing toward the center of the circle, as shown in **figure 3**, asserting that they would meet in the center. As the boys walked, Eric shouted: "Wait! There's a problem. How do I know the steps that John is taking are the same as the steps that Daniel is taking?" Many children agreed with Eric. Stephanie proposed using foot-to-foot steps, which led to further discussion of whose foot to use. Children made cut-outs of their feet to use as measuring units. This exercise resulted in an approximation of the center with which most children were satisfied. Some children thought that perhaps the "Daniel-John Way" should be supplemented by other methods, especially given that the wind made using the foot cutouts problematic.

A second method was proposed. Using a blue rope, two children made a chord that intersected the circle, estimating that the rope passed through the center. Two other children took a white rope to make another chord that intersected the circle, again estimating that the rope passed through the center. These children proposed that the "X" made by the intersecting chords would mark the center if they could coordinate their locations so that each "half" of a chord (the conjectured radius) was the same number of footprints from the center. **Figure 4** shows this strategy. Because the footprints were not precise measures, some children decided to use rulers, although they needed help in interpreting the meaning of parts of units. Through such approximations, the children developed some confidence in their estimates of the location of the center of the circle.

During the next lesson, children again considered the circle. To be more certain about its center, they constructed circles. A few children sprang to the chalkboard and drew circles, but their classmates quickly suggested that their drawings were not adequate. The class agreed

FIGURE 4

Intersecting diameters to find the center of the circle.

that freehand drawings neither defined the center nor ensured equidistance of the points that defined the circle from its center. Recalling the class experience with ropes, Kyle found some string. He tied chalk to one end and held the other end with his finger. He then traced a circle. Some classmates objected, noting that the string stretched. They suggested that using a wooden ruler with holes might be more accurate. This method involved holding the ruler at one end and tracing the circle with a piece of chalk placed in one of the holes. Cassie spoke for the class when she noted that the second tool resulted in circles "without bumps and wobbles," and "with the same distance [gesturing to points on the circle] to the middle."

Model Exploration

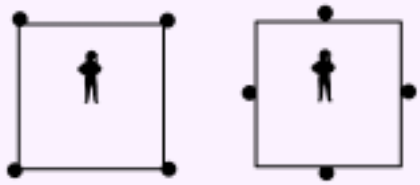
The concluding lesson involved the children in a search for other forms that might be "almost fair" or fair in particular configurations. Leslie, for example, proposed a five-player game that used parts of a square to ensure fairness, as shown in **figure 5**. John manipulated a hexagon-shaped Polydron piece, finding that "six corners would be fair." John noted, "I could draw a circle around the Polydron and a lot of it touches the circle." Penner asked, "What could you do to make [the shape] more fair for more children?" John replied that he would add more sides, and "more corners would touch the circle." The lesson concluded with the children sharing the shapes that they had explored as alternatives to the circle, a shape that they had concluded was "fairest of them all."

Conclusion

The cycles of model invention and revision described here let the children participate in several important processes described in the *Principles and Standards for School Mathematics* (NCTM 2000), especially those concerned with represen-

FIGURE 5

Leslie's fair square



tation and communication. The problem of finding a fair configuration for the game "Mother, may I?" elicited a series of representations of the shape of fairness. The children began with literal configurations of their bodies and were encouraged to use drawings and diagrams to represent these configurations. Using diagrams helped the children think about which aspects of the game were essential for representation and which could be discounted. The game also introduced children to the powerful idea that shapes can be used to model situations. As the children explored the properties of the forms that they selected as models, they used their increasing knowledge of these properties to select new forms that would more closely match their definition of fairness. Thus, distance in the world had its analogue in length measurements of shapes. Properties had a function other than definition.

After discarding other candidates, the children settled on circles as forms that would be good models of fairness in "Mother, may I?" Because their teacher had established classroom norms that required the children to convince others of the validity of their conjectures, the children invented ways to test their conjectures about the properties of circles, rather than simply assert the properties as self-evident. The children invented methods for finding the center of a circle and developed a variety of measurement units to test the property of equidistance of points on the circle from the center. Rather than settle on the circle model, the teacher encouraged the children to consider other forms that might provide reasonable approximations of the shape of fairness. This extension led to further exploration of form and consideration of models as approximations about situations, not truths.

Bibliography

Fennema, Elizabeth, and Thomas A. Romberg, eds. *Mathematics classrooms that promote understanding.* Mahwah, NJ: Lawrence Erlbaum Associates, 1999.

Lehrer, Richard, and Daniel Chazan, eds. *Designing learning environments for developing understanding of geometry and space.* Mahwah, NJ: Lawrence Erlbaum Associates, 1998.

National Council of Teachers of Mathematics. (NCTM). *Principles and standards for school mathematics.* Reston, VA: NCTM, 2000. ▲