Early Childhood Corner

Children's Understanding of Equality: A Foundation for Algebra

Karen P. Falkner, Linda Levi, and Thomas P. Carpenter

Many states and school districts, as well as *Principles and Standards for School Mathematics: Discussion Draft* (NCTM 1998), recommend that algebra be taught in the early childhood years. Although young children often understand much more than traditionally thought, adults can have difficulty conceptualizing what would constitute appropriate algebra for the early childhood years. Fifteen teachers and three university researchers are currently involved in a project to define what algebra instruction can and should be for young children. In this article, we discuss the concept of equality, which is a crucial idea for developing algebraic reasoning in young children.

Misconceptions about the Equals Sign

Even though teachers frequently use the equals sign with their students, it is interesting to explore what children understand about equality and the equals sign. At the start of this project, many teachers asked their students to solve the following problem:

$$8 + 4 = \square + 5$$

At first, this problem looked trivial to many teachers. One sixth-grade teacher, for example, said, "Sure, I will help you out and give this problem to my students, but I have no idea why this will be of interest to you." This teacher found that all twenty-four of her students thought that 12 was the answer that should go in the box. She found this result so interesting that before we had a chance to check back with her, she had the other sixth-grade teachers at her school give this problem to their students. As shown in **table 1**, all 145 sixth-grade students given this problem thought that either 12 or 17 should go in the box.

Why did so many children have trouble with this problem? Clearly, children have a limited understanding of equality and the equals sign if they think that 12 or 17 is the answer that goes in the box. Many young children do, however, understand how to model a situation that involves making things equal. For example, Mary Jo Yttri, a kindergarten teacher, gave her students the problem $4 + 5 = \square + 6$. All the children thought that 9 should go in the box. Yttri then modeled this situation with the children. Together, they made a stack of four cubes, then a stack of five cubes. In another space, they made stacks of nine and six cubes. Yttri asked the children if each arrangement had the same number of

Karen Falkner, kfalkner@madison.k12.wi.us, is a primary-grade teacher at Lapham Elementary School, Madison, WI 53703. She is currently participating in a study of young children's algebraic thinking. Linda Levi, llevi@facstaff.wisc.edu, and Thomas Carpenter, tpcarpen@facstaff.wisc.edu, are affiliated with the Wisconsin Center for Educational Research, Madison, WI 53706. They both study the development of young children's algebraic thinking.

Edited by Kate Kline, kate.kline@wmich.edu, Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008. This column addresses the early childhood teacher's need to support young children's emerging mathematics understandings and skills in a context that conforms with current knowledge about the ways that young children — pre-K-K — learn mathematics. Readers are encouraged to send manuscripts for this section to the editor.

cubes. The children knew that the groupings did not have the same number of cubes and were able to tell her which one had more. Several children were able to tell the teacher how they could make both groupings have the same number of cubes. Even after doing this activity, however, the children still thought that 9 should go in the box in the equation.

This incident surprised Yttri and the researchers. We had assumed that kindergarten children would have little experience with the equals sign and would not yet have formed the misconceptions about equality demonstrated by older children. Even kindergarten children, however, appear to have enduring misconceptions about the meaning of the equals sign that are not eliminated with one or two examples or a simple explanation. This incident also illustrates that children as young as kindergarten age may have an appropriate understanding of equality relations involving collections of objects but have difficulty relating this understanding to symbolic representations involving the equals sign. A concerted effort over an extended period of time is required to establish appropriate notions of equality. Teachers should also be concerned about children's conceptions of equality as soon as symbols for representing number operations are introduced. Otherwise, misconceptions about equality can become more firmly entrenched. (See "About the Mathematics" on p. 234.)

As Behr, Erlwanger, and Nichols (1975); Erlwanger and Berlanger (1983); and Anenz-Ludlow and Walgamuth (1998) have documented, children in the elementary grades generally think that the equals sign means that they should carry out the calculation that precedes it and that the number after the equals sign is the answer to the calculation. Elementary school children generally do not see the equals sign as a symbol that expresses the relationship "is the same as."

Not much variety is evident in how the equals sign is typically used in the elementary school. Usually, the equals sign comes at the end of an equation and only one number comes after it. With number sentences, such as 4 + 6 = 10 or 67 - 10 - 3 = 54, the children are correct to think of the equals sign as a signal to compute.

First and Second Grades

Karen Falkner is currently teaching a first– and second–grade class. Children typically stay in the class for two years. The remainder of this article shows how the children in this class have progressed in their understanding of equality over the past year-and-a-half.

For some time, solving story problems has been an integral part of mathematics instruction in Falkner's class. Students are regularly asked to write number sentences that show how they solved story problems. Falkner expected her students to be successful, therefore, when she first asked them to solve the number sentence 8 + 4 = □ + 5. To her surprise, the students answered the problem just as research indicated that they would. Most put 12 in the box, and some extended the sentence by adding = 17. The discussion that followed was interesting. Most said that 12 should go in the box because "eight plus four equals twelve." The following excerpt illustrates the class discussion that took place after students had worked on the problem.

Falkner. Is 8 + 4 the same as 12 + 5?

Anna. No

Falkner. Then why did you put 12 in the box?

Anna. Because 8 + 4 equals 12. See? [Counting on her fingers] It's 8,9,10,11,12. [Many children nod their heads in agreement.]

Falkner. Did anyone get another answer?

Adam. It is 7.

Falkner. Why?

Adam. Because you have to have the same amount on each side of the equals

TABLE 1 Percent of children offering various solutions to 8 + 4 = □ + 5

Grade	Answers Given: 7	12	17	12 and 17	Other	Number of Children
1	0	79	7	0	14	42
1 and 2	6	54	20	0	20	84
2	6	55	10	14	15	174
3	10	60	20	5	5	208
4	7	9	44	30	11	57
5	7	48	45	0	0	42
6	0	84	14	2	0	145

sign. That's what the equals sign means.

Falkner. I see. Adam, would you say that again?

[*Adam* repeats his explanation. Other children, considering Adam a class leader, listen attentively.]

Falkner. [Gesturing at the number sentence on the chalkboard.] So, Adam, you say that the equals sign means that however much something is on one side of the equals sign, the same amount has to be on the other side of the equals sign. [Looking at the rest of the class] What do you think about what Adam said?

Anna. Yes, but it has to be 12, because that is what 8 + 4 equals.

Dan. No, Adam is right. Whatever is on one side of the equals sign has to equal what is on the other side: 8 + 4 = 12 and 7 + 5 = 12, so 7 goes in the box.

The class wrestled with this problem for some time. The equals sign is a convention, the symbol chosen by mathematicians to represent the notion of equality. Because no logical reason exists that the equals sign does not mean "compute," Falkner thought that it was appropriate to tell the class that she agreed with Adam and Dan. Telling the class what the equals sign meant was not, however, sufficient for many children to be able to adopt the standard use of the sign.

Falkner then chose to develop her students' understanding of the equals sign through discussion of true and false number sentences; this discussion builds on the work of Robert Davis (1964). Falkner presented number sentences, similar to the following, to her students and asked whether the number sentences were true or false.

4 + 5 = 9	12 - 5 = 9	7 = 3 + 4
8 + 2 = 10 + 4	7 + 4 = 15 - 4	8 = 8

The children's reactions were interesting. All agreed that the first sentence was true and that the second was false. They could prove these assertions by a number of means. They were less sure about the remaining sentences.

Falkner. What about this sentence? 7 = 3 + 4. Is it true or false? [Lots of squirming around, distressed faces, and muttering from the class.]

Gretchen. Yes, 3 + 4 does equal 7.

Ned. But the sentence is wrong.

Anna. It's backward.

Falkner. But Adam has told us that the equals sign means that the quantity on each side of it has to be equal. Is that true here?

Anna. Yes, but it's the wrong way.

Falkner. Let's try this. [She models the problem, giving one child seven Unifix

About the Mathematics

Children must understand that equality is a relationship that expresses the idea that two mathematical expressions hold the same value. It is important for children to understand this idea for two reasons. First, children need this understanding to think about relationships expressed by number sentences. For example, the number sentence 7 + 8 = 7 +7 + 1 expresses a mathematical relationship that is central to arithmetic. When a child says, "I don't remember what 7 plus 8 is, but I do know that 7 plus 7 is 14 and then 1 more would make 15," he or she is explaining a very important relationship that is expressed by that number sentence. Children who understand equality will have a way of representing such arithmetic ideas; thus they will be able to communicate and further reflect on these ideas. A child who has many opportunities to express and reflect on such number sentences as 17 - 9 = 17 - 10 + 1 might be able to use the same mathematical principle to solve more difficult problems, such as 45 - 18, by expressing 45 - 18 = 45 - 20 + 2. This example shows the advantages of integrating the teaching of arithmetic with the teaching of algebra. By doing so, teachers can help children increase their understanding of arithmetic at the same time that they learn algebraic concepts.

A second reason that understanding equality as a relationship is important is that a lack of such understanding is one of the major stumbling blocks for students when they move from arithmetic to algebra (Kieran 1981; Matz 1982). Consider, for example, the equation $4x + 27 = 87$. How do you start solving this equation? Your first step probably involves subtracting 27 from 87. Why may we do so? We may do so because we subtract 27 from *both sides* of the equation. If the equals sign signifies a relationship between two expressions, it makes sense that if two quantities are equal, then 27 less of the first quantity must equal 27 less of the second quantity. What about children who think that the equals sign means that they should do something? What chance do they have of being able to understand the reason that subtracting 27 from both sides of an equation maintains the equality relationship? These students can only try to memorize a series of rules for solving equations. Because such rules are not embedded in understanding, students are highly likely to remember them incorrectly and not be able to apply them flexibly. For these reasons, children must understand that equality is a relationship rather than a signal to do something.

cubes in a stack and asking him to stand on one side of her. She gives another child a stack of four Unifix cubes for one hand and a stack of three for the other hand. That child stands on the other side of her.] Now, do these two children have the same number of cubes?

Class. Yes.

Falkner. Does it make any difference which side of me they stand on? [She asks them to change places, which they do.]

Class. No, but….

As you can imagine, the fourth number sentence caused confusion for many children. Some children thought the number sentence was true because 8 + 2 does equal 10. Children who had a firm understanding of equality were able to explain that this number sentence was not true because 8 + 2 is 10 and 10 + 4 is 14 and 10 is not the same as 14.

When Falkner came to the final sentence, 8 = 8, the class was quite disturbed. Anna spoke for the students when she said, "Well, yes, eight equals eight, but you just shouldn't write it that way." In the few remaining weeks of school, Falkner continued to give problems to her students with the equals sign in various locations.

The Next Year

In the fall, Falkner posed the same problem, 8 + 4 = □ + 5, to her class. A few, but not all, of the children who had been in the room the previous spring correctly solved the problem. Many new first graders proudly put 12 in the box; others looked at the sentence in confusion and asked for help. A discussion similar to the one in the spring ensued. This time, however, a few children understood the notion of equality and enthusiastically explained why the number 7 belonged in the box. Lillie gave the most spirited explanation. "The equals sign means that it has to be even. The amount has to be the same on each side of the equals sign. [Gesturing with her hands] It is just like a teeter-totter. It has to be level."

This class discussion was the first of several about similar open number sentences. Each discussion had its doubters, as well as children who once again explained the idea that each side of the equals sign had to "equal" the same amount. As Falkner listened to the discussions, noted who was talking, and looked at facial expressions, it appeared that the children were beginning to grasp this notion of equality but that the concept was not easily or quickly understood. Falkner was convinced that the notion of equality would take time for all the children to understand, and she returned to it often as the year progressed.

Falkner integrated discussion of equality throughout the school year in two ways. First, she continued to present open number sentences in which she varied the location of the unknown. Some examples of these open number sentences included the following: □ = 9 + 5, 7 + 8 = □ + 10, and 7 + □ = 6 + 4. Second, she presented true and false sentences, such as those in the examples, to encourage children to reflect on the meaning of the equals sign. She also had the children write their own true and false number sentences. The tasks that Falkner used to build children's understanding of equality were also tasks that build their understanding of number operations.

As the year progressed, more and more children began to understand equality. In March, the class had the following discussion:

Falkner. Look at this number sentence: 8 + 9 = □ + 10. What should go in the box?

"Yes, eight equals eight, but you just shouldn't write it that way"

Carrie. It should be 17.

Skip. But 8 + 9 would equal 17, so 17 + 10 would equal 27, so 17 isn't OK to put in the box.

Myra. Right; 17 + 10 does not equal 17.

Ned. I think that 7 goes in the box; 7 + 10 is 17 and 8 + 9 is 17. Both sides are even. [The class generally agrees, although Carrie is not yet convinced.]

Falkner. Think about what we know about the equals sign. Look at this number sentence: 4898 + 3 = 4897 + □. Can you figure this one out without even doing the addition?

Larry. I think that 4 goes in the box; 4897 is 1 down from 4898, so you need to add 1 more to 3.

Falkner. Did anyone do it a different way? [Children shake their heads. In general, the class agrees that Larry's way gives the right answer and is easy.]

Such discussions about number sentences gave the children an important context for discussing equality throughout the school year. As the year progressed, discussions about equality became integrated with discussions about other algebraic arithmetic concepts. In the following example, the children discuss a much more sophisticated problem that involves an understanding of variables and operations, as well as equality.

Falkner asked the class to look at the sentence $a = b + 2$. She said that the sentence was true and asked the class which was larger, *a* or *b*? Children who think of the equals sign as a signal to do something would have trouble with this problem. Because 2 is added to *b* and nothing is added to *a*, they might think that *b* is larger. The class first agreed that *a* and *b* were symbols for variables, just as a box or triangle were. The class then quickly agreed that *a* was larger, and their arguments for that position clearly indicate a sophisticated understanding of equality.

Falkner. Why do you think that *a* is larger?

Anna. They split the *b* and 2 apart; *a* brings them together.

Jerry. I think *a* [is larger]. That plus 2 is part of *a*.

Myra. Yes *a* has to be bigger because whatever $b + 2$ is has to be higher than *b* because you combine them.

Anna. Right; *a* has the + 2 in it and *b* doesn't.

Lillie. Together they have to be the same; $b + 2$ has to be the same as *a*.

Conclusion

Discussions such as these, which involved an ever-growing number of children, indicate that the children have learned to see the equals sign as a symbol describing a relationship rather than as a "do it" sign. Because this article was written before the end of the school year, we have not collected summary data on children's understanding of the problem $8 + 4 = \square + 5$ in this class. In a pilot study involving a similar first– and second–grade classroom in the same town, however, we found that at the end of the year, fourteen out of sixteen children correctly answered that 7 should go in the box.

As we reflect on our introduction of the notion of equality and the equals sign to this class and others, we continue to be amazed at the interest and excitement that the children bring to the discussions. Lillie uses her teeter-totter metaphor with the enthusiasm of a child ready to play on one. Skip is genuinely outraged that anyone should fill in a blank so that an equation reads $17 = 27$. These are not the bored comments of children looking forward to recess but the excited contributions of children who are exploring a new world of thinking and communicating mathematically and who are enjoying the power of that new knowledge. These children are developing an understanding of equality as they learn about numbers and operations. This understanding will allow them to reflect on equations and will lay a firm foundation for later learning of algebra.

References

Anenz-Ludlow, Adalira, and Catherine Walgamuth. "Third Graders' Interpretations of Equality and the Equal Symbol." *Educational Studies in Mathematics* 35 (1998); 153-87.

Behr, Merlyn, Stanley Erlwanger, and Eugene Nichols. *How Children View Equality Sentences*. PMDC Technical Report, no. 3. Tallahassee, Fla.: Florida State University, 1975. ERIC No. ED 144 802.

Davis, Robert B. *Discovery in Mathematics: A Text for Teachers*. Reading, Mass.: Addison-Wesley Publishing Co., 1994.

Erlwanger, Stanley, and Maurice Berlanger. "Interpretations of the Equal Sign among Elementary School Children." *In Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Montreal: 1983.

Kieran, Carolyn. "Concepts Associated with the Equality Symbol." *Educational Studies in Mathematics* 12 (August 1981): 317-26.

Matz, Marilyn. "Towards a Process Model for School Algebra Errors." *In Intelligent Tutoring Systems*, edited by Derick Sleeman and John Seeley Brown, 25-50. New York: Academic Press, 1982.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics: Discussion Draft*. Reston, Va.: NCTM 1998.

The research reported in this article was supported in part by a grant from the U.S. Department of Education, Office of Educational Resreach and Improvement, to the National Center for Improving Student Learning and Achievement in Mathematics and Science (R305A60007-98). Opinions expressed do not necessarily reflect the position, policy, or endorsement of the supporting agency.