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A Contextual Investigation of Three-Digit Addition and Subtraction

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DISCUSSIONS of the role of algorithms in the elementary school curriculum give rise to conflicting notions of how algorithms should be approached in the classroom. These views range from encouraging students to invent their own algorithms with minimal guidance to teaching students to perform traditional algorithms. In this paper, we will discuss an approach that eschews both these extremes. This approach values students' construction of nonstandard algorithms. However, it also emphasizes the essential role of the teacher and of instructional activities in supporting the development of students' numerical reasoning. In addition, this approach highlights the importance of discussions in which students justify their algorithms. It therefore treats students' development of increasingly sophisticated algorithms as a means for conceptual learning. In our view, an approach of this type is consistent with reform recommendations, like those of the National Council of Teachers of Mathematics (1989), that stress the need for students to develop what Skemp (1976) calls a relational understanding rather than merely to memorize the steps of standard procedures. The contrast between this

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approach and traditional instruction is captured by the difference between "How can I figure this out?" and "What was I told I was supposed to do?" (Yackel et al. 1990). In the approach we will discuss, relational understanding occurs as teachers systematically support students' construction of personally meaningful algorithms. These algorithms emerge when students engage in sequences of problem-solving activities designed to provide opportunities for them to make sense of their mathematical activity. An immediate consequence of this focus on algorithms as symbolizing numerical relationships is that it precludes the unreasonable answers that often occur from the misapplication of little-understood standard algorithms.

To clarify our viewpoint, we will present episodes taken from a third-grade classroom in which we conducted a nine-week teaching experiment. (The authors of this paper were all members of the research team involved in the teaching experiment. The first author was also the classroom teacher who appears in the episodes in this paper.) One of the goals of the experiment was to develop an instructional sequence designed to support third graders' construction of increasingly sophisticated conceptions of place-value numeration and increasinalv efficient algorithms for adding and subtracting three-digit numbers. Our intent is not to offer examples of exemplary teaching but instead to provide a context in which to examine the relationship between numerical understanding and the use of powerful algorithms. In addition, the episodes will illustrate how students' beliefs about mathematics in school influence their development of personally meaningful algorithms. In the following sections of this paper, we will first outline the instructional sequence developed in the course of the experiment and then focus on the students' algorithms and their beliefs.

THE INSTRUCTIONAL SEQUENCE

The instructional sequence we designed to support the development of understanding and computational facility in an integrated manner centered on the scenario of a candy factory. The activities initially involved Unifix cubes as substitutes for candies and later involved the development of ways of recording transactions in the factory (cf. Cobb, Yackel, and Wood [1992] and Bowers [1995]). During initial whole-class discussions, the students and teacher negotiated the convention that single pieces of candy were packed into rolls of ten and ten rolls were packed into boxes of one hundred. The ensuing activities included tasks in estimating and quantifying designed to support the development of enumeration strategies. These activities involved showing the students drawings of rolls and pieces with an overhead projector and asking them to determine how many candies there were in all. In addition, the students were shown rectangular arrays of individual candies and asked to estimate how many rolls could be made from the candies shown. Instructional activities developed later in the sequence included situations in which the students "packed" and "unpacked" Unifix cubes into bars or rolls of ten.

To help students develop a rationale for these activities, the teacher explained that the factory manager, Mr. Strawberry, liked his candies packed so that he could tell quickly how many candies were in the factory storeroom. In order to record their packing and unpacking activity, the students developed drawings and other means of symbolizing as models of their mathematical reasoning (Gravemeijer 1997). The goal of subsequent instructional activities was then to support the students' efforts to mathematize their actual and recorded packing and unpacking activity so that they could interpret it in terms of the composition and decomposition of arithmetical units. To encourage this process, the teacher elaborated on the students' contributions by describing purely numerical explanations in terms of the packing activity and vice versa. This, in turn, served to support the students' development of situation-specific imagery of the transactions in the factory that would provide a foundation for students' mathematical reasoning throughout the sequence.

In a subsequent phase of the sequence, the students were asked to make drawings to show *different ways* that a given number of candies might be arranged in the storeroom if the workers were in the process of packing them. For example, 143 candies might be packed up into one box and four rolls, with three single pieces, or they might be stored as twelve rolls and twenty-three pieces. When the students first described their different ways, they either drew pictures or used tally marks or numerals to record and explain their reasoning. Later, the teacher encouraged all the students to use numerals. To this end, she introduced an inventory form that was used in the factory to keep track of transactions in the storeroom. The form consisted of three columns that were headed from left to right, "Boxes," "Rolls," and "Pieces" (see fig. 18.1). The issue of how to symbolize these different ways and thus the composition and decomposition of arithmetical units became an explicit topic of discussion and a focus of activity. For example, a typical suggestion for verifying that 143 candies could be symbolized as twelve rolls and twenty-three pieces was to pack twenty pieces into two rolls and then pack ten rolls into a box, with three pieces remaining, as shown in figure 18.1.

As a final phase of the sequence, the inventory form was used to present addition and subtraction tasks in what was, for us, the standard vertical-column format (see fig. 18.2). These problems were posed in the context of Mr. Strawberry's filling orders by taking candies from the storeroom and sending them to shops or by increasing his inventory as workers made more candies. The different ways in which the students conceptualized and symbolized these transactions gave rise to discussions that focused on the students' emerging addition and subtraction algorithms.

Boxes	Rolls	Pieces
1	4	
	~~14~~	3
	~~12~~	~~23~~

Fig. 18.1. The inventory form for the candy factory

Boxes	Rolls	Pieces
3	2	7
+ 2	5	8

Fig. 18.2. An addition problem posed on the inventory form

CLASSROOM EPISODES

Throughout the teaching experiment, the students engaged in problem-solving tasks that focused on transactions in the candy factory. As the sequence progressed, we inferred that most of the students' activity was grounded in the situation-specific imagery of the candy factory. Our primary source of evidence was that the students' explanations appeared to have numerical significance in that they spoke of the quantities signified by drawings and numerals instead of merely specifying how they had manipulated digits. Further, they often referred to conventions in the factory such as packing up in the storeroom. Examples of students' varied yet personally meaningful ways of calculating are shown in figure 18.3.

The nonstandard approach of starting the computation with the boxes (the hundreds column in traditional algorithms) (see, e.g., figs. 18.3a and 18.3d) was used by many of the students and became an acceptable way to solve tasks. Since the goal was *not* to ensure that all the students would eventually use the traditional algorithm, the teacher continued to support the development of solutions that could be justified in quantitative terms to other members of the classroom community. Thus, the focus in discussions was on the numerical meanings that the students' records on the inventory form had for them.

Not until the ninth and final week of the teaching experiment did the issue of where to start when calculating emerge as an explicit topic of conversation. The following problem was posed:

> There are five boxes, two rolls, and seven pieces in the storeroom. Mr. Strawberry sends out an order for one box, four rolls, and two pieces. What is left in the storeroom?

Aniquia offered the first solution, shown in figure 18.4. She explained that she first took two pieces from the seven pieces in the storeroom and that she then unpacked a

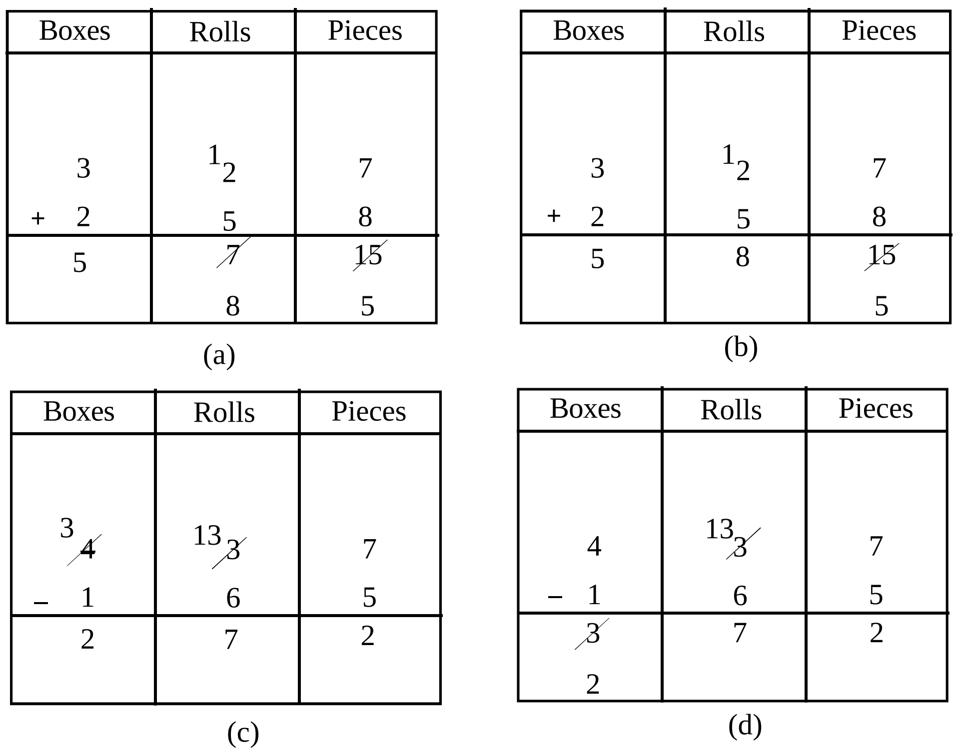

Fig. 18.3. Students' solution to addition and subtraction tasks

box so that she could send out four rolls. The teacher drew pictures of boxes, rolls, and pieces to help other students understand Aniquia's reasoning (see fig. 18.4).

After checking whether the students had any questions for Aniquia, the teacher asked, "Did anybody do it a different way?" Bob raised his hand and shared his solution method, which involved sending out a box first (see fig. 18.5). After Bob had finished his explanation, the teacher asked, "All right. Now, Bob said I'm going to send out my box before I send out my rolls, and Aniquia said I'm going to send out my rolls before I send out my box. Does it matter?"

Some of the students responded:

Cary: It depends on the kind of problem it is.

Avery: Nope, you can do it either way.

Ann: When you usually do subtraction you always start at the right 'cause the pieces are like the ones.

Avery: But it doesn't matter.

Boxes	Rolls	Pieces
4 ~~5~~	12 ~~2~~	7
− 1	4	2
3	8	5

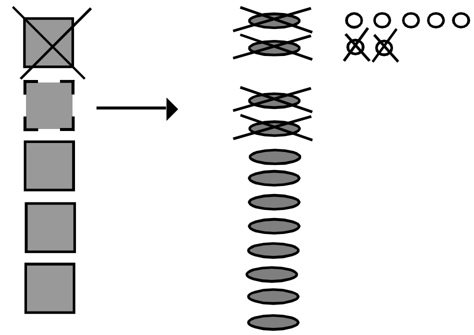

Fig. 18.4. Aniquia's solution on the inventory form and the teacher's graphical explanation

Rick: Yeah, 'cause we're in the candy factory, not in the usual stuff.

At the end of this session, the project team reflected on the students' discussions about where to start and agreed that it would be beneficial to revisit the issue. We thought that many of the students were making the decision on the basis of what they knew about the standard algorithm. The students were third graders, and they had received two years of traditional instruction and had been taught the standard algorithms for adding and subtracting two-digit numbers. It appeared from the students' conversations that they had agreed it did not matter because they were in the candy factory and "not in the usual stuff." We speculated that Cary's comment about depending on the kind of problem referred to whether or not it was posed in the context of the candy factory. For these students there appeared to be two different "maths"– regular school mathematics and the mathematics they did with us in the setting of the candy factory. They were therefore able to justify the difference in where to start because of the different norms and expectations in the two instructional situations. In addition, not only did the students seem to believe that the two "maths" had conflicting norms, but they also appeared to hold different beliefs about what constituted acceptable explanations and justifications in each situation. Explanations in the candy factory focused on imagined numerical transactions in the store-

Boxes	Rolls	Pieces
5	12 ~~2~~	7
− 1	4	2
~~4~~ 3	8	5

Fig. 18.5. Bob's solution on the inventory form

room and on how they could be expressed on the inventory form. School mathematics entailed explanations of symbol manipulations that need not have quantitative significance.

The next day, the following problem was posed to the class:

> There are three boxes, three rolls, and four pieces of candy in the storeroom. Mr. Strawberry gets an order for two boxes, four rolls, and one piece. How many candies are left in the storeroom after he sends out his order?

The first solution was offered by Martin, who completed the task by first drawing boxes, rolls, and pieces on the white board and then recording the result on the inventory form (see fig. 18.6). It is important to note that his description of his solution process was grounded in the imagery of the candy factory, as evidenced by his explanation, which entailed comments such as "First, I sent out the two boxes."

Boxes	Rolls	Pieces
3	3	4
− 2	4	1
0	9	3

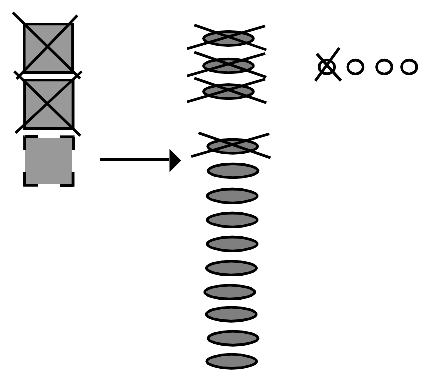

Fig. 18.6. Martin's solution and graphical explanation

When Martin had finished, the teacher asked if the students had questions or comments for Martin. Lenny responded that the inventory form did not provide a record of Martin's unpacking actions in the simulated storeroom and that he needed to show that he had unpacked a box. At this point, the teacher decided to take advantage of this reference to notation by reintroducing the question of "where to start." To reflect Lenny's suggestion, she marked Martin's form as though he had first sent out his pieces (see fig 18.7a). Next, she wrote the same problem on the board again, this time recording a solution in which the boxes were sent out first, as Martin had described (see fig. 18.7b). The following discussion ensued:

Teacher: Does it matter? Is it OK either way?

Rick: No.

Teacher: But we get the same answer.

Rick: But it's confusing. It's harder to work it out. It's easier that way (he points to the form in fig. 18.7a).

Teacher: But my question is, Do we have to do pieces, then rolls, then boxes, or can we do the boxes first?

Bob: No.

Avery: You can start with boxes or rolls.

Rick: Yeah, but that's confusing.

Teacher: OK, let me tell you what I'm hearing you say. I'm hearing you say that it's easier and less confusing [to start with pieces].

Ann: I think you can start with boxes or pieces but not the rolls.

Bob: It would be harder to start with rolls.

Avery: Yeah.

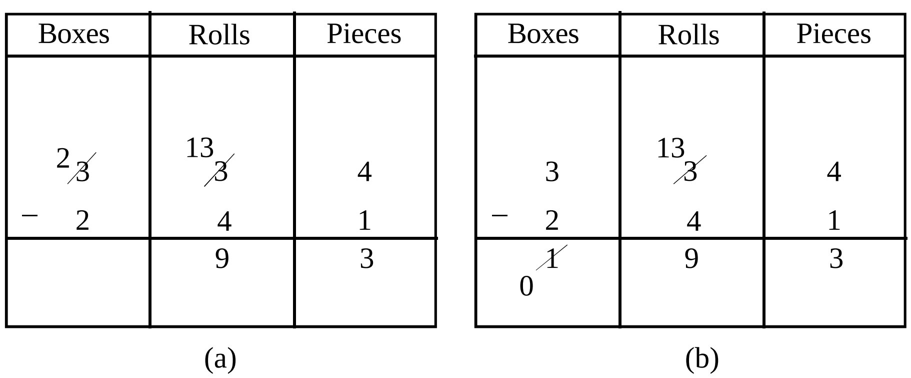

Fig. 18.7. Two subtraction solutions

At this point, the discussion shifted to focusing on what was an easier or a less confusing way to solve the problem. The students appeared to agree that both processes were legitimate and that their emphasis was on the ease of comprehension. The teacher's questions were not intended to steer the students to the standard algorithm. Instead, the students' judgments of which approach was easier to understand was made against the background of their prior experiences of doing mathematics in school.

After these discussions, we posed tasks in the traditional column format without the inventory form and discussed the differences in the formats:

Teacher: This is kind of different from the inventory form.

Jess: It's kinda different 'cause it doesn't say boxes, rolls, pieces.

D'Metrius: You still don't have to have boxes, rolls, pieces. It don't have to be up there 'cause you know these are pieces, rolls, and boxes.

Jess: I just know that . . . I remember boxes, rolls, and pieces help me.

It appeared from the conversation that many of the students evoked the imagery of the candy factory as they interpreted and solved these tasks. This conjecture was corroborated by the observation that as they worked in pairs on activity sheets, many of the students continued to explain and justify their solutions in terms of activity in the candy factory. In some instances, the students actually drew boxes, rolls, and pieces to support their solutions. The many personally meaningful ways of solving the tasks that emerged are shown in figure 18.8. These included adding from the left (fig. 18.8a), adding from the right (fig. 18.8b), and working from a drawing and recording the result (fig. 18.8c).

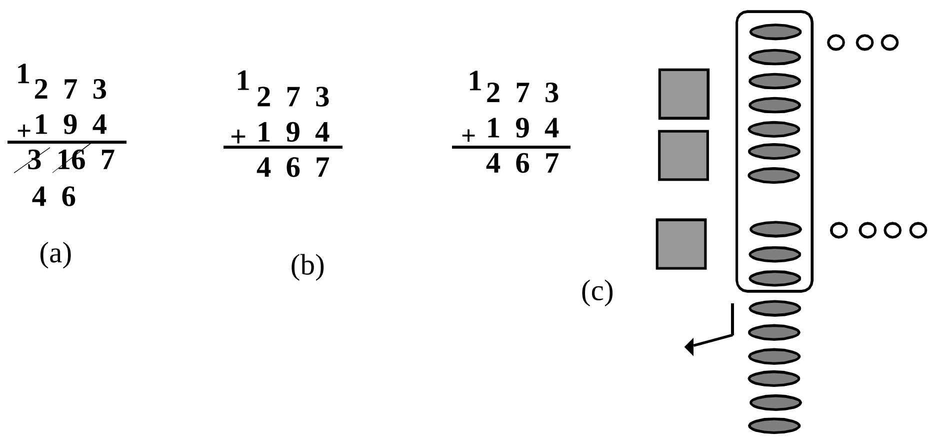

Fig. 18.8. Three different solutions to the problem 273 + 194

CONCLUSION

We have outlined an instructional sequence designed to support students' construction of their own personally meaningful algorithms for three-digit addition and subtraction as it was conducted in one third-grade classroom. Although the situation-specific imagery of the candy factory appeared to support most of the students' activity, the influence of their prior participation in the practices of school mathematics became apparent when they discussed which types of solutions were legitimate and easier to comprehend. However, they did seem to modify their views about what it means to know and do mathematics (at least when interacting with McClain as the teacher) as they compared and contrasted solutions. The important norm that became established was that of explaining and justifying solutions in quantitative terms. We find this significant because the students were able to reorganize their beliefs about doing mathematics in this setting. This change in beliefs could be characterized, using Skemp's (1976) distinction, as shifting from instrumental views toward more relational views of doing mathematics. A comparison of interviews conducted with the students before and after the classroom teaching indicated that they had all made significant progress in their numerical reasoning (Bowers 1996).

The episodes discussed in this paper provide support for a change in emphasis "toward conjecturing, inventing, and problem solving-away from an emphasis on mechanistic answer-finding" (NCTM 1991, p. 3). This emphasis should in no way be construed to mean that students do not need to construct powerful algorithms; it simply calls attention to the importance of students' developing increasingly sophisticated numerical understandings as they develop personally meaningful algorithms. We have illustrated one attempt to achieve this goal that emphasizes the teacher's proactive role, the contribution of carefully sequenced instructional activities, and the importance of discussions in which students explain and justify their thinking.

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